Série TD N° 03 (TF)

* Exo1 :

1) Par utilisation des deux propriétés :

on a :

$$TF\{x(t)\, e^{j2\pi f_0 t}\} = X(f - f_0).$$

$$TF\{1\} = \delta(f).$$

Donc :

$$TF\{e^{j2\pi f_0 t}\} = \delta(f - f_0).$$

2) $s(t) = \cos(2\pi f_0 t)$

$$= \frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2} = \frac{1}{2}\left[e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right]$$

$$TF\{s(t)\} = TF\left\{\frac{1}{2}\left[e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right]\right.$$

$$= \frac{1}{2} TF\left\{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right]$$

$$= \frac{1}{2}\left[TF\{e^{j2\pi f_0 t}\} + TF\{e^{-j2\pi f_0 t}\}\right.$$

$$\boxed{S(f) = \frac{1}{2}\left[\delta(f - f_0) + \delta(f + f_0)\right]}$$

Exo 2:

a) $x(t) = e^{at}\,\mathcal{E}(-t)\ ;\ a>0$.

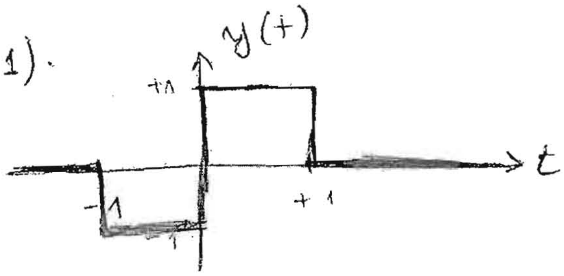
$$X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt\ ;\quad X(f) = TF\{x(t)\}.$$

$$= \int_{-\infty}^{+\infty} e^{at}\,\mathcal{E}(-t)\, e^{-j2\pi f t}\, dt = \int_{-\infty}^{0} e^{at}\, e^{-j2\pi f t}\, dt.$$

$$= \int_{-\infty}^{0} e^{(a-j2\pi f)t}\, dt = \left[\frac{e^{(a-j2\pi f)t}}{a-j2\pi f}\right]_{-\infty}^{0} = \frac{1-0}{a-j2\pi f}.$$

$$\Rightarrow \boxed{X(f) = \frac{1}{a-j2\pi f}}$$

b) $y(t) = -\mathcal{E}(t+1) + 2\mathcal{E}(t) - \mathcal{E}(t-1)$.



$$Y(f) = TF\{y(t)\}.$$

$$= \int_{-\infty}^{+\infty} y(t)\, e^{-j2\pi f t}\, dt$$

$$= \int_{-1}^{0} (-1)\, e^{-j2\pi f t}\, dt + \int_{0}^{1} (1)\, e^{-j2\pi f t}\, dt$$

$$= -\left[\frac{e^{-j2\pi f t}}{-j2\pi f}\right]_{-1}^{0} + \left[\frac{e^{-j2\pi f t}}{j2\pi f}\right]_{0}^{1}$$

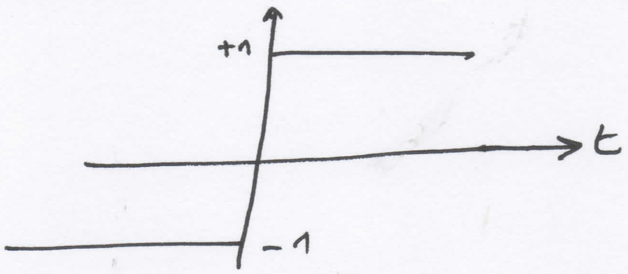
$$= \frac{1 - e^{j2\pi f} + 1 - e^{-j2\pi f}}{j2\pi f} = \frac{2 - 2\cos(2\pi f)}{j2\pi f}.$$

$$= \frac{2(1-\cos(2\pi f))}{2j\pi f} = \frac{1-\cos(2\pi f)}{j\pi f}$$

*Exo 3 :

$$x(t) = \begin{cases} +1 & t \geq 0 \\ -1 & t < 0 \end{cases} = \mathrm{sgn}(t).$$

1)° Représentation de $x(t)$:



2)° La Transformée de Fourier :

En regardant dans la table des TF :

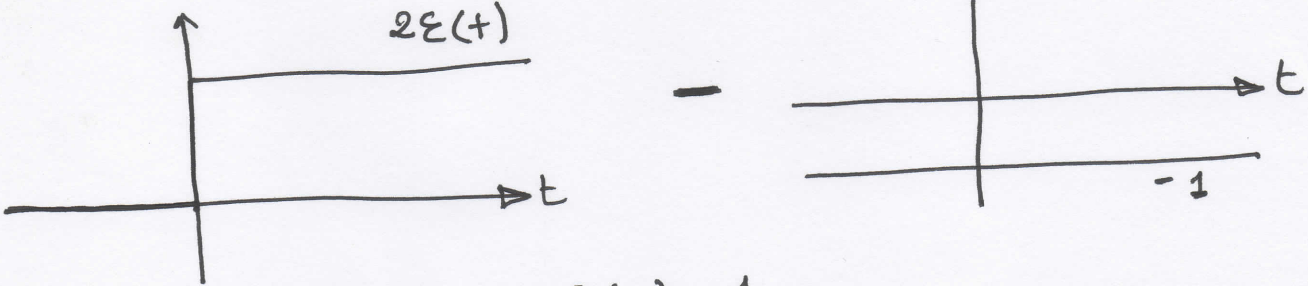
$$\mathrm{sgn}(t) \xrightarrow{TF} \frac{1}{j\pi f} \quad \text{au sens des distributions.}$$

3)° Par calcul :

En exploitant la properiété de dérivation :

$$\frac{d^n x(t)}{dt^n} \xrightarrow{TF} (j2\pi f)^n X(f).$$

or : on peut écrire autrement le signal $x(t)$:



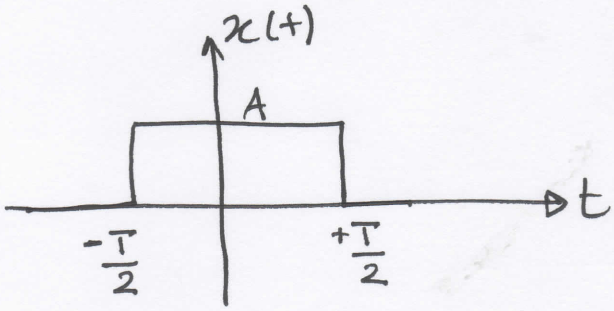
Donc : $x(t) = 2\varepsilon(t) - 1$.

$\frac{dx(t)}{dt} = 2\delta(t)$ et la $TF\{2\delta(t)\} = 2$.

Donc : $2 = j2\pi f\, X(f) \Rightarrow \boxed{X(f) = \frac{1}{j\pi f}}$

* Exo 4 :

$$x(t) = A\,\mathrm{Rect}\left(\frac{t}{T}\right)$$



T ≠ période
c'est la largeur de la porte.

1) La TF :

$$X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt = \int_{-\frac{T}{2}}^{+\frac{T}{2}} A\, e^{-j2\pi f t}\, dt$$

$$X(f) = \frac{-A}{j2\pi f}\left[e^{-j2\pi f t}\right]_{-\frac{T}{2}}^{+\frac{T}{2}}$$

$$= \frac{-A}{2j\pi f}\left[e^{-j\pi f T} - e^{+j\pi f T}\right] = \frac{A}{\pi f}\left[\underbrace{\frac{e^{j\pi f T} - e^{-j\pi f T}}{2j}}_{\sin(\pi f T)}\right]$$

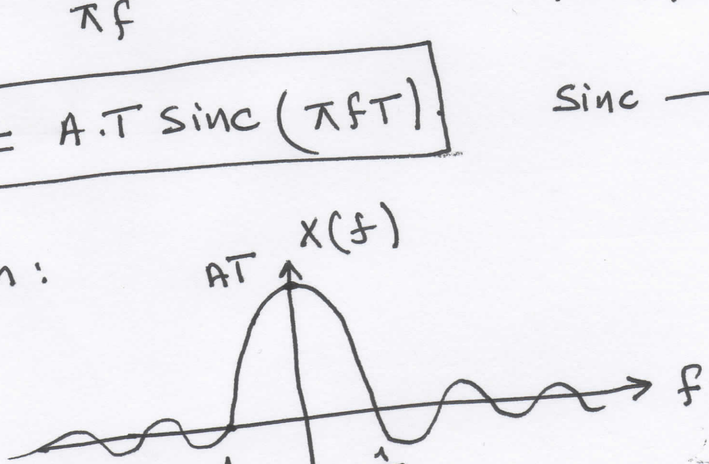
Donc :

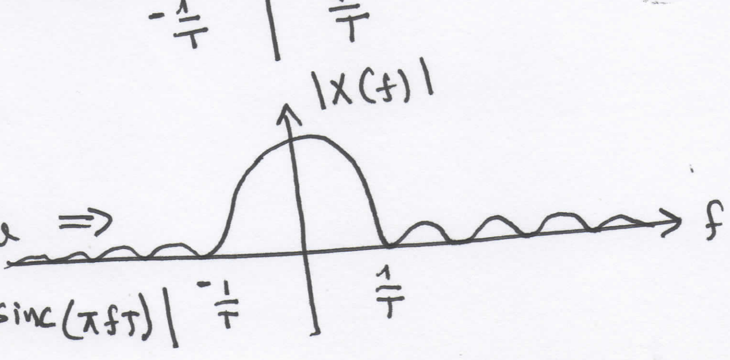
$$X(f) = \frac{A}{\pi f}\cdot \sin(\pi f T) = \frac{T}{T}\cdot\frac{A}{\pi f}\sin(\pi f T).$$

$$\boxed{X(f) = A.T\,\mathrm{sinc}(\pi f T)}$$

sinc ⟶ sinus cardinal.

2) Representation :





Spectre d'amplitude ⟹

$|X(f)| = |AT\,\mathrm{sinc}(\pi f T)|$

3) Densité spectrale d'énergie : DSE

$$\Gamma(f) = |X(f)|^2 = (AT)^2 \operatorname{sinc}^2(\pi f T).$$

4)° Energie :

~~suite~~ : $E = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt$ (temps).

$E = \int_{-\infty}^{+\infty} |X(f)|^2 \, df$ (fréq).

$$E = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt = \int_{-\frac{T}{2}}^{+\frac{T}{2}} A^2 . \, dt \Rightarrow \boxed{E = A^2 T}$$

$$E = \int_{-\infty}^{+\infty} |X(f)|^2 \, df = \int_{-\infty}^{+\infty} (AT)^2 \operatorname{sinc}^2(\pi f T) \, df.$$

$$= A^2 T \underbrace{\int_{-\infty}^{+\infty} T \operatorname{sinc}^2(\pi f T) \, dt}_{= 1 \text{ (propriété)}}.$$

Donc :

$$\boxed{E = A^2 T}$$

Donc le Théorème de Parseval est vérifié :

$$\int_{-\infty}^{+\infty} |x(t)|^2 \, dt = \int_{-\infty}^{+\infty} |X(f)|^2 \, df$$