TD ≠ 4

Polynômes de Taylor et Séries Entières

Exercice 1

ⓐ $f(x)=\frac{1}{x}$, $a=2$, $n=4$

$f(x)=\frac{1}{x}$ $\qquad f(a)=f(2)=\frac{1}{2}$

$f'(x)=-\frac{1}{x^2}=-x^{-2}$ $\qquad f'(a)=f'(2)=-\frac{1}{4}$

$f''(x)=2x^{-3}=\frac{2}{x^3}$ $\qquad f''(a)=f''(2)=\frac{1}{4}$

$f'''(x)=-6x^{-4}=-\frac{6}{x^4}$ $\qquad f'''(a)=f'''(2)=-\frac{3}{8}$

$f\ (x)=24x^{-5}=\frac{24}{x^5}$ $\qquad f^{(4)}(a)=f^{(4)}(2)=\frac{3}{4}$

$$P_4(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\frac{f^{(4)}(a)}{4!}(x-a)^3$$

$$P_4(x)=\frac{1}{2}-\frac{1}{4}(x-2)+\frac{1}{8}(x-2)^2-\frac{1}{16}(x-2)^3+\frac{1}{32}(x-2)^4$$

ⓑ $f(x)=\frac{1}{1+x}$, $a=0$, $n=5$

$f(x)=(1+x)^{-1}=\frac{1}{1+x}$ $\qquad f(0)=1$

$f'(x)=-(1+x)^{-2}=\frac{-1}{1+x}$ $\qquad f'(0)=-1$

$f''(x)=2(1+x)^{-3}=\frac{2}{(1+x)^3}$ $\qquad f''(0)=2$

$f'''(x)=-6(1+x)^{-4}=\frac{-6}{(1+x)^4}$ $\qquad f'''(0)=-6$

$f^{(4)}(x)=24(1+x)^{-5}=\frac{24}{(1+x)^5}$ $\qquad f^{(4)}(0)=24$

$$P_5(x)=f(0)+f'(0).x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{(4)}}{4!}x^4$$

$$P_5(x)=1-x+x^2-x^3+x^4$$

ⓒ $f(x) = \sin(\pi x)$, $a = 0$, $n = 3$

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$f(x) = \sin(\pi x)$ $\qquad$ $f(a) = f(0) = 0$

$f'(x) = \pi\cos(\pi x)$ $\qquad$ $f'(a) = f'(0) = \pi$

$f''(x) = -\pi^2\sin(\pi x)$ $\qquad$ $f''(a) = f''(0) = 0$

$f'''(x) = -\pi^3$ $\qquad$ $f'''(a) = f'''(0) = -\pi^3$

$$P_3(x) = 0 + \pi x + 0 \cdot - \frac{\pi^3}{3!}x^3$$

$$P_3(x) = \pi x - \frac{\pi^3}{6}x^3$$

Exercice 2

Maclaurin

$f(x) = \cos x$ $\qquad$ $f(0) = 1$

$f'(x) = -\sin x$ $\qquad$ $f'(0) = 0$

$f''(x) = -\cos x$ $\qquad$ $f''(0) = -1$

$f'''(x) = \sin x$ $\qquad$ $f'''(0) = 0$

$f^{(4)}(x) = \cos x$ $\qquad$ $f^{(4)}(0) = 1$

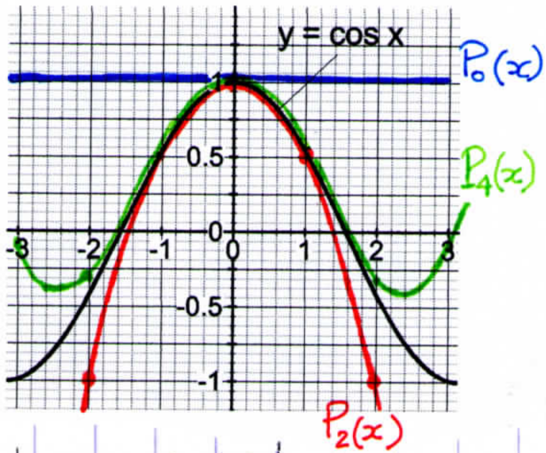
$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots + \frac{f^{(n)}(0)}{n!}x^n$$

$P_0(x) = f(0) = 1$ $\qquad$ $P_0(x) = 1$

$P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = 1 - \frac{x^2}{2}$ $\qquad$ $P_2(x) = 1 - \frac{x^2}{2}$

$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 = 1 - \frac{x^2}{4} + \frac{x^4}{4}$

x	-3	-2	-1	0	1	2	3
$\cos x$	-0.99	-0.42	0.54	1	0.54	-0.42	-0.99
$P_0(x)$	1	1	1	1	1	1	1
$P_2(x)$	-3.5	-1	0.5	1	0.5	-1	-3.5
$P_4(x)$	-0.125	-0.33	0.54	1	0.54	-0.33	-0.125



$P_1(x) = P_0(x) = 1$ car le coefficient $f'(0)$ du terme en x est nul.

Exercice 3

1. $f(x) = \cos x \quad , \quad x = 0.3 \quad , \quad n = 4$

$$\cos(0.3) \simeq P_4(0.3) = 1 - \frac{(0.3)^2}{2!} + \frac{(0.3)^4}{4!} = 0{,}995{,}337{,}5$$

L'erreur de l'approximation est

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1} \qquad \xi \in [0, x] \text{ ou } [x, 0]$$

$$R_4(0.3) = \frac{f^{(5)}(\xi)}{5!}(0.3)^5 = \frac{\sin \xi}{5!}(0.3)^5 \leq \frac{(0.3)^5}{5!} = 0.000{,}02$$

2. $f(x) = \ln(1+x) \quad , \quad a = 0 \quad , \quad x = 0.5$

$$\text{erreur} = \frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1} \quad , \quad \xi \in [0, 0.5]$$

$$f(x) = \ln(1+x) \longrightarrow f'(x) = (1+x)^{-1}$$
$$\longrightarrow f''(x) = -(1+x)^{-2} \longrightarrow f'''(x) = 2(1+x)^{-3}$$
$$\longrightarrow f^{(4)}(x) = -6(1+x)^{-4} \longrightarrow f^{(5)}(x) = 24(1+x)^{-5}$$
$$\dots \longrightarrow f^{(n)}(x) = (n-1)!\,(1+x)^{-n}(-1)^{n+1}$$

$$\text{erreur} = R_n(x) = \left| \frac{(n-1)!\,(1+\xi)^{-n}(-1)^{n+1}}{(n+1)!} x^{n+1} \right|$$

$$= \frac{x^{n+1}}{(1+\xi)^n n(n+1)} \leq \frac{x^{n+1}}{n(n+1)} = \frac{0.5^{n+1}}{n(n+1)} \leq 0.5^{n+1} \leq 0.0001$$

$$\longrightarrow \ln(0.5^{n+1}) \leq \ln 0.0001$$

$$\longrightarrow (n+1)\ln(0.5) \leq \ln(0.0001)$$

$$\longrightarrow n+1 \geq \frac{\ln 0.0001}{\ln 0.5} \longrightarrow n \geq 1 + \frac{\ln 0.0001}{\ln 0.5} = 14.28$$

Car $\ln(0.5) < 0$

On peut donc prendre $\boxed{n = 15}$

TD#4 ex3

③ $f(x) = \sin x \; , \; a = 0 \; , \; n = 3$

$$P_3(x) = x - \frac{x^3}{3!}$$

$$R_4(x) = \left| \frac{f^{(5)}(\xi)}{5!} x^5 \right| = \left| \frac{\cos \xi}{5!} x^5 \right| \leq \frac{|x|^5}{120} \leq 0.01 \Rightarrow |x| \leq (1.2)^{\frac{1}{5}}$$

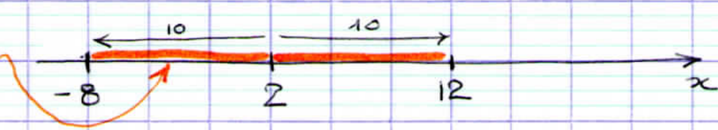
$$\rightarrow |x| \quad 1.037 \longrightarrow -1.037 \leq x \leq 1.037$$

Exercice 4

ⓐ $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{10^n}$

$a_n = \dfrac{1}{10^n} \qquad \dfrac{1}{R} = \lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \lim\limits_{n\to\infty} \dfrac{10^n}{10^{n+1}} = \dfrac{1}{10} \rightarrow R = 10$

la serie converge dans



- Cas de $\boxed{x = -8}$

  la serie devient $\displaystyle\sum_{n=1}^{\infty} \frac{(-10)^n}{10^n} = \sum_{n=1}^{\infty} (-1)^n$ diverge $\left(\begin{array}{l} \text{test du terme } n \\ \lim\limits_{n\to\infty} a_n \neq 0 \end{array} \right)$

- Cas de $\boxed{x = 12}$

  la serie devient $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{10^n} = \sum_{n=1}^{\infty} 1 = 1+1+1+\cdots$ div (test du terme n)

donc l'intervalle de convergence est $(-8, 12)$

ⓑ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n!} \longrightarrow a_n = \frac{(-1)^n}{n!}$

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(n+1)!} \cdot \frac{n!}{(-1)^n} \right| = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n} = 0$$

donc $R = \infty$ la serie converge dans $(-\infty, \infty)$

ⓒ $\displaystyle\sum_{n=1}^{\infty} (2n)! \left(\frac{x}{2}\right)^n \qquad a_n = \frac{(2n)!}{2^n}$

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{[2(n+1)]!}{2^{n+1}} \cdot \frac{2^n}{(2n)!} \right| = \lim_{n\to\infty} \frac{(2n+2)!}{(2n)!} \cdot \frac{1}{2}$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{2} = \infty \Rightarrow R = 0, \text{ la serie diverge } \forall x \neq 0$$

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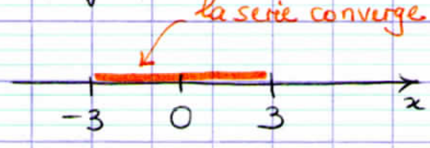
d $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n\sqrt{n}\, 3^n}$ $\qquad a_n = \dfrac{1}{n\sqrt{n}\,3^n}$

$$\frac{1}{R} = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n\sqrt{n}\,3^n}} = \lim_{n\to\infty} \frac{1}{3\sqrt[n]{n^{3/2}}}$$

$$= \lim_{n\to\infty} \frac{1}{3\,(\underbrace{\sqrt[n]{n}}_{1})^{3/2}} = \frac{1}{3} \Longrightarrow \boxed{R=3}$$



$x=3$ la série s'écrit $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ Converge (série de Riemann $p=\frac{3}{2}>1$)

$x=-3$ la série s'écrit $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n}}$ cv car ABSOLUMENT CONVERGENTE

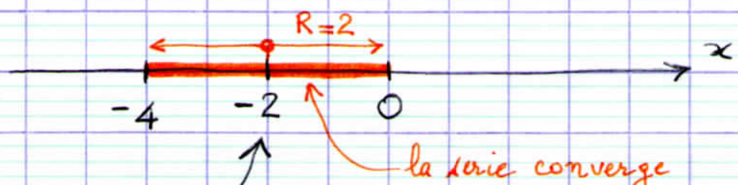
donc l'intervalle de convergence est $[-3, 3]$

e $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x+2)^n}{n\,2^n}$ $\qquad a_n = \dfrac{(-1)^{n+1}}{n\,2^n}$

$$\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n\,2^n}{(n+1)\,2^{n+1}} = \lim_{n\to\infty} \frac{1}{2}\cdot\frac{n}{n+1} = \frac{1}{2} \Rightarrow \boxed{R=2}$$

$x+2 = (x-(-2))$

$a=-2$



$\boxed{x=-4}$ la série s'écrit $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-2)^n}{n\,2^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-1)^n(2^n)}{n\,2^n}$

$$= \sum_{n=1}^{\infty} \frac{(-1)(-1)^{2n}}{n\,2^n} = -\sum_{n=1}^{\infty} \frac{1}{n}$$ diverge série harmonique

$\boxed{x=0}$ la serie s'ecrit $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}2^n}{n\,2^n} = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n}$ Converge d'après le test of Series Alternées

donc l'intervalle de convergence est $(-4, 0]$

$$\text{(f)} \quad \sum_{n=1}^{\infty} \frac{1+2+3+\dots+n}{1^2+2^2+3^2+\dots+n^2} x^n$$

on sait que $1+2+3+\dots+n = \dfrac{n(n+1)}{2}$

et $1^2+2^2+3^2+\dots+n^2 = \dfrac{n(n+1)(2n+1)}{6}$ $\Big\} \Rightarrow a_n = \dfrac{\cancel{n(n+1)}}{2} \times \dfrac{6}{\cancel{n(n+1)}(2n+1)} = \dfrac{3}{2n+1}$

$$\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{3}{2(n+1)+1} \times \frac{2n+1}{3} = \lim_{n\to\infty} \frac{2n+1}{2n+3} = 1 \Rightarrow R=1$$

la serie converge pour $-1 < x < 1$

- $\boxed{x=1}$ la serie s'écrit $\displaystyle\sum_{n=1}^{\infty} \frac{3}{2n+1}(1)^n = \sum_{n=1}^{\infty} \frac{3}{2n+1} = 3 \sum_{n=1}^{\infty} \frac{1}{2n+1}$

$a_n = f(n)$ avec $f(x) = \dfrac{1}{2x+1}$ $f \geq 0$, decroissante et

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{dx}{2x+1} = \frac{1}{2}\int_1^{\infty} \frac{2}{2x+1}\,dx = \frac{1}{2} \lim_{X\to\infty} \int_1^X \frac{2}{2x+1}\,dx$$

$$= \frac{1}{2} \lim_{X\to\infty} \left[\ln(2x+1)\right]_1^X = \frac{1}{2} \lim_{X\to\infty} \left(\ln(2X+1) - \ln 3\right) = \infty$$

donc l'integrale diverge $\Longrightarrow$ la serie diverge d'aprè le test de l'integrale

- $\boxed{x=-1}$ la série s'ecrit $\displaystyle\sum_{n=1}^{\infty} \frac{3}{2n+1}(-1)^n = 3\sum_{n=1}^{\infty} (-1)^n \frac{1}{2n+1}$ CV test des series alternés

donc l'intervalle de convergence est $[-1, 1)$

$$\text{(g)} \quad \sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2} x^n$$

$$\frac{1}{R} = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left(\left(\frac{n}{n+1}\right)^{n^2}\right)^{1/n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$$

$$= \lim_{n\to\infty} \left(\frac{n+1-1}{n+1}\right)^n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right)^n = \lim_{n\to\infty} \frac{\left(1-\frac{1}{n+1}\right)^{n+1}}{\left(1-\frac{1}{n+1}\right)} = \frac{e^{-1}}{1} = \frac{1}{e}$$

$\Rightarrow \boxed{R=e}$ la serie converge pour $x \in (-e, e)$

- $\boxed{x=e}$ $\longrightarrow$ la série s'écrit $\displaystyle\sum_{n=1} \left(\frac{n}{n+1}\right)^{n^2} e^n$

...

- $\boxed{x=-e}$

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