MAS209: Fluid Dynamics

DAVID BURGESS AND HENK VAN ELST
Astronomy Unit
School of Mathematical Sciences
Queen Mary, University of London
Mile End Road
London E1 4NS
United Kingdom
E–Mail: D.Burgess@qmul.ac.uk, H.van.Elst@qmul.ac.uk

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MAS209: Fluid Dynamics

- These lecture notes are only intended for the use of students studying the course MAS209: Fluid Dynamics. Any distribution for any other purpose is not permitted.

- The selection of topics and material covered in these lecture notes is influenced by the following textbooks, lecture notes and and websites:

  - *Elementary Fluid Dynamics*, D J Acheson, Claredon Press, Oxford, 1990.
  - *A First Course in Fluid Dynamics*, A R Paterson, Cambridge University Press, Cambridge, 1983.
  - *Fluid Dynamics*, R Tavakol, Queen Mary, University of London, Lecture notes, 1980ies. Unavailable.
  - *Fluid Mechanics*, L D Landau and E M Lifshitz, Pergamon Press, Oxford, 1959.
  - *Theorie B: Elektrodynamik*, Gottfried Falk, University of Karlsruhe, Notes taken in lectures, Summer Semester 1988 [in German]. Unavailable.
  - *Physik III: Thermodynamik*, Friedrich Herrmann, University of Karlsruhe, Script, 1997 [in German]. Available online at URL: `www-tfp.physik.uni-karlsruhe.de/~didaktik/`.
  - *The MacTutor History of Mathematics archive*, URL: `www-history.mcs.st-andrews.ac.uk`
  - Wolfram Research's *World of Science*, serviced by E W Weisstein, URL: `scienceworld.wolfram.com`

- Good starting points for the interested reader to obtain access to pictures, animated material and current research in Fluid Dynamics are provided by the websites:

  - *CFD Online*, URL: `www.cfd-online.com`
  - *eFluids: A Free One-Stop Resource For Fluid Dynamics and Flow Engineering*, URL: `www.efluids.com`
  - *arXiv.org e-Print archive – Fluid Dynamics*, research papers, URL: `arxiv.org/list/physics.flu-dyn/recent`

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Describing fluids and fluid flows

1.1 Introduction

We are all familiar with **fluids**: water, coffee, air, treacle, and so on. And we have all been fascinated at some time by fluid behaviour: pouring, splashing, swimming, and so on. At some level we have all built up an intuitive understanding of how fluids behave. However, there are still fluid phenomena which are vital to modern life but which most people do not understand; the best example is aircraft flight.

Understanding **fluid dynamics** has been one of the major advances of **physics**, **applied mathematics** and **engineering** over the last hundred years. Starting with the explanations of **aerofoil theory** (i.e., why aircraft wings work), the study of fluids continues today with looking at how **internal** and **surface waves**, **shock waves**, **turbulent fluid flow** and the occurrence of **chaos** can be described mathematically. At the same time, it is important to realise how much of engineering depends on a proper understanding of fluids: from flow of water through pipes, to studying effluent discharge into the sea; from motions of the atmosphere, to the flow of lubricants in a car engine. **Fluid dynamics** is also the key to our understanding of some of the most important phenomena in our physical world: ocean currents and weather systems, convection currents such as the motions of molten rock inside the Earth and the motions in the outer layers of the Sun (cf. the lectures on MAS211 Introduction to Stellar Structure), the explosions of supernovae, the swirling of gases in galaxies, and, when combined with Einstein's relativistic theory of gravitation, modelling the evolution of the observable part of the Universe (cf. the lectures on MAS313 Cosmology).

Like many fascinating subjects, understanding is not always easy. In particular, for **fluid dynamics** there are many terms and mathematical methods which will probably be unfamiliar. Although the basic concepts of velocity, mass, linear momentum, forces, etc., are the building blocks, the slippery nature of fluids means that applying those basic concepts sometimes takes some work. Fluids can fascinate us exactly because they sometimes do unexpected things, which means we have to work harder at their mathematical explanation.

In this first chapter we will introduce most of the terms which are used when describing **fluids** and **fluid flows**. These terms will be used throughout the course, and we need to explain them before we start any mathematical description of **fluid dynamics**. We will only discuss *non-relativistic* fluid flows, where fluid and propagation speeds are much less than the speed of light.

1.2 Wood, water, air

A block of wood, a bottle of water, a container of air: all these have some similarities. They are definite amounts of substance with definite boundaries. But they are also very different, since the materials have very different properties: water and air can be poured, but wood cannot; air needs a container which completely contains it, whereas water will find its own shape.

From the point of view of a mathematical description they are all examples of what is called a **continuous medium**: a substance whose structure is continuous, rather than discrete.

Now let us examine some similarities and differences:

- An obstacle can be inserted into water and air and moved around without irreversibly dividing the continuous medium — water and air are examples of fluids.

- Water and air both need containers (i.e., boundaries of some sort) but water can also have a surface with no fixed external boundary (a free boundary). Wood has its own boundaries and has no need for imposed boundaries. Water is thus an example of a liquid (i.e., can have a free surface), but air is an example of a gas.

- Both air and water can be made to flow: they can be pushed and their shape deforms as they are in motion. **Fluid dynamics** is the study of such flows — what is possible and what is not.

- Air and water respond differently to applied external forces. Squeezing air can fairly easily change its volume (think of squeezing a bottle of air), but this is not true of water. This is a property related to the compressibility of the fluid. Gases are generally more compressible than liquids. When we compress some fluid we decrease the volume it occupies, but its mass remains constant so that its density (mass per unit volume) increases. Thus a compressible fluid changes density as it gets squashed and squeezed. But if a fluid (for example, such as water) does not change its density when it is squashed, then it is called incompressible.

- Now consider water and treacle: Both are liquids but there are clearly some major differences between them, and the one that is most noticeable is that of viscosity. We say that treacle is an example of a viscous fluid (with a high viscosity), whereas water has a low (but non-zero) viscosity. When we put a knife into a fluid and move it in the plane of the blade so that we are trying to slip the knife through the fluid rather than pushing the fluid around, then it is clear that it is more difficult to move the knife in treacle than in water. In other words, treacle offers more resistance to this shearing motion (i.e., motion in the plane of the knife) than water. Indeed this is the fundamental definition of viscosity.

1.3 Forces: motion and equilibrium

1.3.1 MKS-system of physical dimensions

Throughout this course the **physical dimensions** of any physical quantity will be expressed exclusively in terms of the **MKS-system**.[1] Here the basic physical dimensions are

$$[\,\text{length}\,]\ , \quad [\,\text{mass}\,]\ , \quad [\,\text{time}\,]\ , \quad [\,\text{temperature}\,]\ ,$$

with **SI units** 1 m, 1 kg, 1 s, and 1 K. The last of these units is named after the Irish mathematician and physicist William Thomson Kelvin (1824–1907).

To give some of the most prominent examples, the physical dimensions of **velocity**, **acceleration**, **linear momentum**, **force**, **energy** and **entropy** correspond to

$$\begin{aligned}
[\,\text{velocity}\,] &= [\,\text{length}\,]\,[\,\text{time}\,]^{-1} \\
[\,\text{acceleration}\,] &= [\,\text{length}\,]\,[\,\text{time}\,]^{-2} \\
[\,\text{lin. momentum}\,] &= [\,\text{mass}\,]\,[\,\text{length}\,]\,[\,\text{time}\,]^{-1} \\
[\,\text{force}\,] &= [\,\text{mass}\,]\,[\,\text{length}\,]\,[\,\text{time}\,]^{-2} \\
[\,\text{energy}\,] &= [\,\text{mass}\,]\,[\,\text{length}\,]^{2}\,[\,\text{time}\,]^{-2} \\
[\,\text{entropy}\,] &= [\,\text{mass}\,]\,[\,\text{length}\,]^{2}\,[\,\text{time}\,]^{-2}\,[\,\text{temperature}\,]^{-1}\ .
\end{aligned}$$

The physical dimension of any other quantity that arises in this course can be constructed in a similar fashion.

1.3.2 Newton's equations of motion

For an object such as a "particle" of some sort, of **mass** m, we know that we can describe changes in its motion by the mechanical laws of motion that the English physicist and mathematician Isaac Newton (1642-1727) introduced in 1687 (cf. the lectures on MAS107 Newtonian Dynamics and Gravitation). For example, if there is a **force** $\boldsymbol{F}$ acting on a particle with **velocity** $\boldsymbol{v}$, then **Newton's equations of motion**

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \boldsymbol{F}(\boldsymbol{p},\boldsymbol{r}) \qquad\qquad \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \boldsymbol{v}(\boldsymbol{p},\boldsymbol{r}) \tag{1.1}$$

describe the rate of change in time of the particle's **linear momentum** $\boldsymbol{p}$ and its **position** $\boldsymbol{r}$, respectively. When m is *constant*, so that $\boldsymbol{p}$ is simply related to $\boldsymbol{v}$ by $\boldsymbol{p} = m\,\boldsymbol{v}$ and $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{r})$ only, the first can be re-written as

$$m\,\boldsymbol{a} = \boldsymbol{F}\ , \tag{1.2}$$

where $\boldsymbol{a} = \mathrm{d}\boldsymbol{v}/\mathrm{d}t = \mathrm{d}^2\boldsymbol{r}/\mathrm{d}t^2$ is the particle's **acceleration**. From knowledge of $\boldsymbol{F}$ we can solve the particle's **equations of motion**, and, with its **initial position** and **initial velocity** *given*, we can find out *where* the particle goes and

[1] This name derives from the units "metre", "kilogram" and "second".

in *what* time. We also know that if there is a balance of forces on the object, then it will *not* change its velocity; in particular, if its velocity is initially zero, then it will remain at rest. Thus, **Newton's equations of motion** allow us to describe both motion and equilibrium.

The study of **fluids** is not so simple, because a fluid is extended over space, and when a fluid moves, e.g., because of motion of its **boundaries**, then forces are communicated to the interior of the fluid *by the fluid itself*. However, underlying all of **fluid dynamics** are the empirically verified physical principles of **conservation of mass**, **conservation of energy** and **conservation of linear momentum**, combined with the **laws of thermodynamics**.[2] We will see in this course how they are applied in order to derive the equations which govern fluid motion.

When we consider forces on a **continuous medium**, then we have to consider forces which act to move or rotate the body, and in addition we have to consider forces which change the volume and shape of the material. **Elastic solids** are another example of a continuous media; e.g., a block of foam plastic. We can study the elastic properties of such a material, i.e., how the shape changes when various kinds of forces are applied. There are different kinds of such forces: squeezing (compression) and shearing. Compression acts to change the volume of the material. Shearing acts to change the shape of the material (e.g., from a cube to a parallelepiped).

Solids change their shape and deform until a balance is reached between the applied force and internal forces (or the material breaks — end of continuous medium!). **Fluids**, on the other hand, will deform continuously (and keep changing shape) under the action of a **shearing force**. This is the *fundamental definition* of a **fluid**.

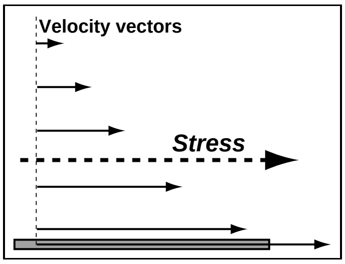
Sometimes the boundary between solid and liquid can be fuzzy (e.g., glass, jelly, powders), with apparent solids behaving like liquids over very long times, and collections of grains of solids having some behaviour which is reminiscent of a fluid.

[2]Unfortunately, many textbooks try to simplify this issue by stating that the laws of fluid dynamics are nothing but an application of Newton's laws of motion. However, conceptually such a statement blurs the issue rather than helps clarifying it.

1.3.3 Viscosity

A fluid differs from a solid in that a fluid *never* comes into equilibrium with a shearing force. The best way to imagine a shearing force is a thin, planar piece of material (e.g., a knife) moving through a fluid in the same plane (i.e., a "cutting" motion). Unlike a solid, if a given force is applied to the plate it will keep on moving through the fluid, although different fluids will oppose its motion by different amounts. In treacle the motion will be slower than in water, for the same applied force. This resistance of the fluid to shearing forces is called **viscosity**.

Imagine some geometrical surface within a fluid (not to be confused with the free surface of a liquid). In a viscous fluid there can be a **stress** (i.e., force per unit area) which is **tangential** to the surface. Consider treacle again: A knife moving through treacle causes motion in the same direction as the knife, even far from the knife, and not because it is being pushed out of the way by the knife. Since the knife is moving in its own plane, the motion can only be communicated by stresses tangential to the knife. For so-called **Newtonian viscous fluids** *the shear stress is proportional to the velocity gradient*, and the constant of proportionality is called the **coefficient of (shear) viscosity**.



Note that the faster moving layer exerts a stress on the slower moving layer in the direction of motion. And, the slower moving layer exerts an equal and opposite stress on the faster layer. So the fast material tries to speed up the adjacent slower material, and the slow material tries to slow down the adjacent faster material. If all the fluid is moving at the same velocity, then there is no velocity shear, and no viscous stresses within the fluid.

If a fluid has a **velocity gradient**, then we say that the flow has shear, or that it is sheared; in other words, there is *relative motion* of fluid elements within the fluid.

For many applications viscosity can often be ignored, i.e., assumed to be zero, and in this case the fluid is described as **inviscid**. Often in **fluid dynamics** one makes the assumption of inviscid flow. But crucial differences in fluid flow appear when a real fluid with viscosity is studied (as compared with the case of an "ideal fluid" with identically zero viscosity). In particular we shall see that the type of boundary conditions that must be applied will differ for inviscid fluids and for viscous fluids (even when the viscosity tends to zero).

Some liquids do not have a Newtonian viscosity, but rather one where the shear stress is related to the fluid velocity gradient in a *non-linear* fashion (e.g., non-drip paint, blood, printing inks). They exhibit many interesting aspects, but they are out of the scope of this course.

1.4 Describing a fluid

1.4.1 Material properties

Water and treacle can both be poured from a spoon, but they obviously have some properties that make them different. These are properties of the fluid, rather than the particular fluid flow at any instant, so that we may think of them as constant parameters describing the fluid.

- **Chemical composition**: What type of fluid is it? Usually this has an obvious answer (water or treacle), but can sometimes be complicated if the fluid is undergoing a chemical reaction (e.g., combustion is a fluid dynamic phenomenon).

- **Viscosity**: Treacle is more viscous than water. A lot of **fluid dynamics** is concerned with inviscid flow, but the role of viscosity is crucial to understanding some of the most important fluid phenomena, such as lift produced by a wing.

- **Equation of state**: This is a functional relation between the pressure and density (and often temperature) which is characteristic of the type of fluid, and which expresses the elastic properties of the fluid, i.e., how easily it can be squashed. It thus describes how the density of the fluid changes in response to changes in pressure and temperature. This introduces the concept of **compressibility**: sponge is more easily compressed than steel.

We shall see that important distinctions can be made between **inviscid**, **viscous**, and **very viscous flows**.

A further important distinction arises from whether a fluid is **compressible** or **incompressible**. A uniform, incompressible fluid has the simple equation of state $\rho = \mathrm{constant}$, i.e., constant density so that the density remains the same irrespective of how much the pressure is increased. For example, for most purposes water can be regarded as incompressible.

1.4.2 Fluid variables

In order to describe fluid flows, we need to be able to deal with characteristic fluid properties which are different at different spatial positions and times. Mathematically we model this situation with variables that describe the physical state of a fluid usually as functions of time and spatial position. As such the **mathematical model** built in **fluid dynamics** is based on the **continuum hypothesis**: we assume that in a fluid domain $D \subset \mathbb{R}^3$ of Euclidian space we can assign **fluid variables** to any spatial position $\boldsymbol{r}$ at any time t that vary *continuously* and may be taken as constant across sufficiently small volumes. The continuum hypothesis implies that fluid variables are differentiable, and so **fluid dynamics** can be formulated as a **classical field theory**.

The fluid variables represent either scalar-valued or vector-valued fields (in general: tensor-valued fields).

- **Mass density**: The scalar-valued function

$$\rho = \rho(t, \boldsymbol{r})$$

describes the mass density in a given fluid at any time t at any position $\boldsymbol{r}$. It can be defined as

$$\rho = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V} ,$$

where Δm is the mass of a small volume ΔV of the fluid.[3] In the MKS-system mass density has physical dimension $[\,\text{mass}\,][\,\text{length}\,]^{-3}$, i.e., unit mass per unit volume. Its SI unit is thus 1 kg m^{-3}.

- **Flow velocity**: The vector-valued function

$$\boldsymbol{u} = \boldsymbol{u}(t, \boldsymbol{r})$$

describes the flow velocity of a given fluid at any time t at any position $\boldsymbol{r}$. The *main task* of every fluid dynamics problem is to determine $\boldsymbol{u}(t, \boldsymbol{r})$ from the fluid equations of motion for known acting forces. In Cartesian coordinates $\boldsymbol{u}$ might be written componentwise as $\boldsymbol{u} = (u_x, u_y, u_z)^T$, or in some textbooks $\boldsymbol{u} = (u, v, w)^T$. In the MKS-system flow velocity has physical dimension $[\,\text{length}\,][\,\text{time}\,]^{-1}$, i.e., unit length per unit time. Its SI unit is thus 1 m s^{-1}.

- **Pressure**: The force exerted by an **inviscid fluid** is always locally at right angles, i.e., **normal**, to any surface. The pressure at a point is thus the same in all directions, or **isotropic**. For a geometrical **surface element** $\boldsymbol{n}\,\mathrm{d}A$ within the fluid with **unit normal vector** $\boldsymbol{n}$ that force is $p\,\boldsymbol{n}\,\mathrm{d}A$, where

$$p = p(t, \boldsymbol{r})$$

is a scalar-valued function denoting the pressure in a given fluid at any time t at any position $\boldsymbol{r}$, which is independent of $\boldsymbol{n}$. This is the force exerted *on* the fluid into which $\boldsymbol{n}$ is pointing *by* the fluid on the other side of $\mathrm{d}A$. In the MKS-system pressure has physical dimension $[\,\text{mass}\,] \times [\,\text{length}\,]^{-1}[\,\text{time}\,]^{-2}$, corresponding to unit force per unit area. Its SI unit is thus 1 kg m^{-1} s^{-2}, which has been given the name 1 Pa after the French mathematician and philosopher Blaise Pascal (1623–1662).

[3] In reality, ΔV cannot really go to zero since eventually Δm would have to vary discontinuously due to the fact that the fluid is made from individual molecules. However, depending on the size of the molecules in the liquid or gas to be investigated, for practical purposes it is generally sufficient if ΔV approaches the order of 10^{-18} m^3 to 10^{-30} m^3.

For **viscous fluids** the forces exerted across a surface element in the fluid are more complex, having **tangential** as well as **normal components**. But we shall see that we can still define a pressure.

- **Temperature**: The scalar-valued function

$$T = T(t, \boldsymbol{r})$$

describes the temperature in a given fluid at any time t at any position $\boldsymbol{r}$, and is a measure of the internal energy of the fluid, i.e., the energy associated with the thermal motions of the molecules making up the fluid. The SI unit for temperature is 1 K. In some cases one can treat a fluid as having constant, uniform temperature, and in this case the fluid is *isothermal*. Considering equations of state that include temperature would force us to also consider the **laws of thermodynamics** which govern the exchange of thermal energy within the fluid. These matters will *not* be discussed in this course, but they have important implications for the theory of viscous compressible flows.

1.5 Describing fluid flow

1.5.1 Static/dynamic and steady/unsteady

- **Static/dynamic**: If a fluid is at rest everywhere (i.e., $\boldsymbol{u} = \boldsymbol{0}$ everywhere), then the situation is **static**. In this case the appropriate theory is called **hydrostatics**, and we will see some examples of this. The majority of this course is devoted to the study of fluid flow, i.e., where the fluid is **dynamic**. And hence the term “**fluid dynamics**”.

- **Steady/unsteady**: If the fluid flow parameters are functions of space but *not* functions of time, then the flow is described as **steady**, or *time stationary*. Mathematically this is expressed by partial derivatives with respect to time of any fluid variable f being identically zero,

$$\frac{\partial f}{\partial t} \equiv 0 \; . \tag{1.3}$$

In contrast, if the flow velocity changes in time, so that terms involving $\partial/\partial t$ are non-zero, then the situation is termed **unsteady**.

1.5.2 Analysing fluid motions

One way to model fluid flow is to simply investigate flows which have certain properties, without directly worrying about the fluid equations of motion. Usually such properties relate to *differential functions of the fluid velocity*. Useful properties are **divergence-free** or **solenoidal** fluid flows where

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \,, \tag{1.4}$$

and **curl-free** or **irrotational** fluid flows where

$$\boldsymbol{\nabla} \times \boldsymbol{u} = \mathbf{0} \,. \tag{1.5}$$

The aim is then to find solutions for $\boldsymbol{u}$ which satisfy one or the other, or both, of these partial differential equations, together with any boundary conditions. Of course, this would be pointless if the physical principles governing fluid flow meant that the equations were not appropriate. But the fact is that these two differential relations relate to actual physical behaviour: the first to incompressible flow, and the second to flow with zero vorticity (which as we will see is often satisfied). Of course, we will also have to face the possibility that in a given situation the flow *cannot* be described by such differential equations.

Example: steady constant uniform flow

"Steady" means that there is no time dependence of any fluid variable f, i.e., $\partial f/\partial t = 0$; and "uniform" means that there are no spatial gradients. So, if the flow is in the x-direction of a **Cartesian coordinate system**, with magnitude $u_0 =$ constant, then

$$\boldsymbol{u} = (u_0, 0, 0)^T \,. \tag{1.6}$$

And, obviously, it follows that the flow is divergence-free (i.e., solenoidal)

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \,, \tag{1.7}$$

and curl-free (i.e., irrotational)

$$\boldsymbol{\nabla} \times \boldsymbol{u} = \mathbf{0} \,. \tag{1.8}$$

It might seem a trivial result, but it turns out later to be quite interesting that **uniform flow** has zero curl!

Example: planar shear flow

"Planar" means that $\boldsymbol{u}$ is constant in parallel planes; and "shear" means that $\boldsymbol{u}$ varies in some direction. For example, consider a flow which is everywhere in the x-direction of a **Cartesian coordinate system**, but whose magnitude increases with y-position, such as

$$\boldsymbol{u} = (u_0 y, 0, 0)^T \,, \tag{1.9}$$

where u_0 is a constant. This is an example of a *linear* **shear flow**, since the flow velocity depends linearly on spatial position.

For this flow we have zero divergence,

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = \frac{\partial}{\partial x}(u_0 y) = 0 \,, \tag{1.10}$$

but

$$\boldsymbol{\nabla} \times \boldsymbol{u} = (0, 0, -\frac{\partial}{\partial y}(u_0 y))^T = -\, u_0 \, \boldsymbol{e}_z \,, \tag{1.11}$$

so that the curl of the flow is constant and in the z-direction. (We will look at this flow again in the section on vorticity.)

Example: spherical radial outflow

For this example we use **spherical polar coordinates** so that $\boldsymbol{u} = (u_r, u_\vartheta, u_\varphi)^T$. We choose one particular form of **radial outflow**,

$$\boldsymbol{u} = (u_0/r^2, 0, 0)^T \,, \tag{1.12}$$

where u_0 is a constant. Then, using standard formulae for the divergence and the curl of a vector field in spherical polar coordinates (given in chapter 2),

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}(u_\vartheta \sin\vartheta) + \frac{1}{r\sin\vartheta}\frac{\partial u_\varphi}{\partial\varphi} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{u_0}{r^2}) = 0 \,, \tag{1.13}$$

and

$$\boldsymbol{\nabla} \times \boldsymbol{u} = \frac{1}{r^2 \sin\vartheta} \begin{vmatrix} \hat{\boldsymbol{e}}_r & r\hat{\boldsymbol{e}}_\vartheta & r\sin\vartheta\,\hat{\boldsymbol{e}}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\vartheta} & \frac{\partial}{\partial\varphi} \\ u_r & r u_\vartheta & r\sin\vartheta\, u_\varphi \end{vmatrix} = (0, 0, 0)^T \,. \tag{1.14}$$

So, for this particular form of the flow velocity (i.e., for $|\boldsymbol{u}| \propto r^{-2}$), the flow has zero divergence and zero curl.

Example: axisymmetric (azimuthal) flow

Consider circular (azimuthal) flow which is constant on cylinders centred on the z-axis, but whose magnitude depends on the distance from the z-axis. This situation is best described using **cylindrical polar coordinates** so that $\boldsymbol{u} = (u_r, u_\varphi, u_z)^T$. In purely **axisymmetric flow** $u_r = u_z = 0$, and we choose a particular form for u_φ:

$$\boldsymbol{u} = (0, u_0 r^n, 0)^T \,, \tag{1.15}$$

where n is an integer and u_0 is a constant. Then,

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\varphi}{\partial\varphi} + \frac{\partial u_z}{\partial z} = 0 \,, \tag{1.16}$$

so that the flow has zero divergence. The curl of the flow is

$$\boldsymbol{\nabla}\times\boldsymbol{u} = \frac{1}{r}\begin{vmatrix} \hat{\boldsymbol{e}}_r & r\hat{\boldsymbol{e}}_\varphi & \hat{\boldsymbol{e}}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ u_r & ru_\varphi & u_z \end{vmatrix} = (0, 0, \frac{1}{r}\frac{\partial}{\partial r}(u_0 r^{(n+1)})^T = (n+1)u_0 r^{(n-1)}\,\hat{\boldsymbol{e}}_z \,. \tag{1.17}$$

The last result is rather interesting, since one can see that when $n = 1$, then $\boldsymbol{\nabla}\times\boldsymbol{u} = \boldsymbol{0}$, and so the flow is in fact constant uniform, even though all the fluid is rotating around the z-axis.

1.5.3 Vorticity

As well as the flow velocity itself, it is useful to define the **vorticity** of a fluid flow which is equal to the curl of the flow velocity.

The **vorticity** is a vector-valued function of position and time defined as

$$\boldsymbol{\omega} := \boldsymbol{\nabla}\times\boldsymbol{u} \,, \tag{1.18}$$

and it is crucially important in the study of **fluid dynamics**.

The vorticity at a point is a measure of the **local rotation**, or **spin**, of a fluid element at that point. Note that the local spin is *not* the same as the **global rotation** of a fluid.

If the flow in a region has zero vorticity, then the flow is described as **irrotational**. **Irrotational flow** is one of the major categories of fluid flows.

In 2-D flow, in the x–y plane of a **Cartesian coordinate system**, the velocity has the form

$$\boldsymbol{u} = [u_x(t,x,y), u_y(t,x,y), 0]^T \,, \tag{1.19}$$

and the vorticity is $\boldsymbol{\omega} = (0, 0, \omega_z)^T$, where

$$\omega_z = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \,. \tag{1.20}$$

In such a 2-D flow consider two short fluid line segments AB (length δx in the x-direction) and AC (length δy in the y-direction). The y-component of the velocity of B exceeds that of A by approximately

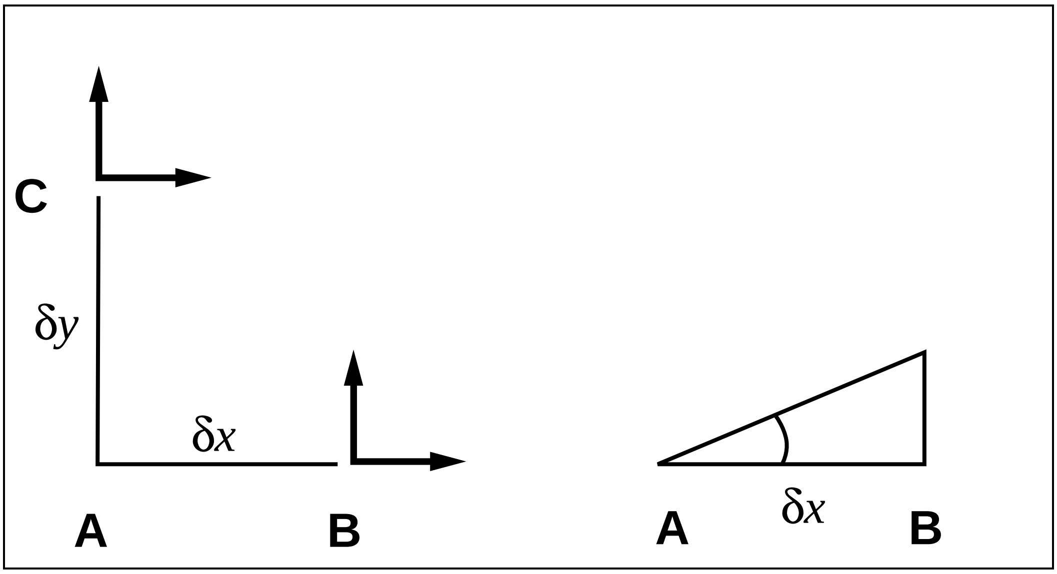
$$\frac{\partial u_y}{\partial x}\,\delta x \,, \tag{1.21}$$

so that in a short time δt B moves relative to A in the y-direction by a distance $\frac{\partial u_y}{\partial x}\delta x\delta t$. This motion then subtends an angle at A of $\frac{\partial u_y}{\partial x}\delta x\delta t/\delta x$ (using $\tan\alpha \approx \alpha$, for small α).

We conclude that the instantaneous angular velocity of AB about the z-direction is simply $\frac{\partial u_y}{\partial x}$. Similarly, the instantaneous angular velocity of AC about the z-direction *in the opposite sense* is $\frac{\partial u_x}{\partial y}$. Thus at any point

$$\frac{1}{2}\,\omega_z = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) \tag{1.22}$$

represents the *average angular velocity of two short fluid line elements which are perpendicular*.



Consider in a **Cartesian coordinate system** the **shear flow**

$$\boldsymbol{u} = (\beta y, 0, 0)^T \ , \tag{1.23}$$

where β is a constant. The fluid is not rotating globally, but has a vorticity

$$\omega_z = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} = -\,\beta \ . \tag{1.24}$$

This arises because while a fluid line segment in the x-direction remains in the x-direction, a line segment initially in the y-direction leans increasingly towards the x-direction, so that there is a non-zero average angular velocity.

1.5.4 Circulation

In order to characterise the large scale rotational properties of the flow we introduce the concept of **circulation**. Let C be some closed curve in the fluid region. Then the circulation around C is defined as the scalar-valued quantity

$$\Gamma := \oint_C \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{l} \ . \tag{1.25}$$

Now, from Stokes' integral theorem (cf. section 2.5 below) we can write

$$\Gamma = \int\!\!\int_S (\boldsymbol{\nabla} \times \boldsymbol{u}) \cdot \mathrm{d}\boldsymbol{A} = \int\!\!\int_S \boldsymbol{\omega} \cdot \mathrm{d}\boldsymbol{A} \ , \tag{1.26}$$

where S is any surface entirely in the fluid that spans C. This might be interpreted as meaning that a flow which is irrotational ($\boldsymbol{\omega} = 0$) will always result in zero circulation. But here care must be taken that there are no obstructions which prevent the surface S being entirely in the flow region. If there is an obstruction in the flow, then Stokes' integral theorem cannot be applied to a circuit enclosing the obstacle, and then there is the possibility of non-zero circulation around the obstacle, even if the vorticity of the flow is zero (i.e., irrotational flow). It turns out that this is very important for understanding how aircraft fly!

1.5.5 Fluid element

When we consider the motion of massive point particles (footballs, rockets, planets, etc.) it is fairly easy to think about the single object which moves in response to the forces acting on it. Fluids, on the other hand, are slippery: they just slip through our fingers! It is much more difficult to think about what it is that is being pushed around, and how the different parts of the fluid interact to produce the complex behaviour that we see. We try to get around this by thinking about an *infinitesimally small volume of fluid* within the whole body of fluid. This **fluid element** is small enough that we can almost consider it as a single point particle (i.e., it does not split up to go around an obstacle), but it still retains the properties of a fluid (i.e., it can continuously deform its shape in response to the forces on it). The concept of an infinitesimal fluid element which moves with the fluid is equivalent to a piece of debris being carried along in the flow of a river. Its motion traces out a particle path, as discussed below. We can say that a particular fluid element is "labelled" by its spatial position $\boldsymbol{q}$ at some given fixed time t.

When we refer to a fluid element, we do not mean one particular molecule of a real fluid (which will have random thermal motions), but rather a particle (real or conceptual) which has the flow velocity $\boldsymbol{u}(t, \boldsymbol{r})$ when it is at spatial position $\boldsymbol{r}$ at time t.

The fluid element moves with the fluid, and as it does so, its shape might distort; in particular, until one of its dimensions is no longer infinitesimal. But, of course, the distinction between this single fluid element and the rest of the fluid is completely arbitrary, and at any time we can choose to consider another "typical" fluid element. In order to describe the behaviour of the fluid as a whole, we use the concept of a fluid element to find out how this typical part of the fluid moves at one particular position and time. By considering the forces acting on the fluid element we are able to find the equations which govern the fluid motion.

Sometimes we will wish to discuss the motion of not just an infinitesimal fluid element, but a finite "**blob**" consisting of the same fluid elements. Such a blob will move and change shape, and is sometimes referred to as a "dyed" region (although the dye is purely imaginary), and will be a dynamic region in space and time as compared to a region fixed in space.

1.5.6 Streamlines, particle paths and streaklines

The flow velocity is the basic description of how a fluid moves in time and space, but in order to visualize the **flow pattern** it is useful to define some other properties of the flow. These definitions correspond to various experimental methods of visualizing fluid flow.

Streamlines

A **streamline** is, at any particular time $t =$ constant, a curve which has the same **direction** as $\boldsymbol{u}(t, \boldsymbol{r})$ at every point $\boldsymbol{r}$. Equivalently, the flow velocity is tangential to a streamline at all points along it. A streamline can be defined passing through every point in the fluid where the flow velocity is non-zero. Streamlines give information about the direction of fluid flow, but *not* about the magnitude of the flow velocity.

A streamline can be viewed as a parametrised curve, $\boldsymbol{r} = \boldsymbol{r}(s)$. As such we obtain it as a solution to the ordinary differential equation of first-order

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s} = \lambda\, \boldsymbol{u}(t, \boldsymbol{r})\big|_{t=\mathrm{const}} \;, \tag{1.27}$$

subject to specifying a single point through which it passes (i.e., giving an "initial condition"); λ is an arbitrary constant. This relation expresses the fact that for $t =$ constant the tangent vector to a streamline and the flow velocity are parallel to each other. In **Cartesian coordinates** it translates to solving the three ordinary differential equations of first-order

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \lambda\, u_x\big|_{t=\mathrm{const}} \;, \quad \frac{\mathrm{d}y}{\mathrm{d}s} = \lambda\, u_y\big|_{t=\mathrm{const}} \;, \quad \frac{\mathrm{d}z}{\mathrm{d}s} = \lambda\, u_z\big|_{t=\mathrm{const}} \;. \tag{1.28}$$

Equivalently, we can solve the three partial differential equations of first-order

$$u_y(t, \boldsymbol{r})\mathrm{d}z = u_z(t, \boldsymbol{r})\mathrm{d}y \;, u_z(t, \boldsymbol{r})\mathrm{d}x = u_x(t, \boldsymbol{r})\mathrm{d}z \;, u_x(t, \boldsymbol{r})\mathrm{d}y = u_y(t, \boldsymbol{r})\mathrm{d}x \;, \tag{1.29}$$

at any particular time $t =$ constant. This is just the statement that the components of the tangent vector to a streamline are in the same proportion to each other as the components of the flow velocity. This defines a whole family of curves. Any particular curve is chosen by specifying one point through which it passes.

By drawing, or imagining, many streamlines in a fluid (each passing through distinct sets of points) we obtain a map of the motion of the fluid as a whole.

Streamlines can be imagined in a number of ways. Consider long, thin, grass-like water weeds in a stream: at any moment a strand of weed shows the changes in the direction of velocity at a point. From these directions a streamline can be constructed.

Alternatively, there is the experimental technique of a streak photograph. Small, neutrally buoyant beads are put into the fluid, and then one particular plane of the fluid region is illuminated, and a short exposure time photograph is taken. As the fluid moves it carries the beads along with it, and the photograph will record each bead as a streak, the length and direction of which gives the velocity at that particular point in space. (Obviously, the exposure time has to be long enough for the beads to move a short distance). The photograph then gives a view of the streamlines in the illuminated plane.

Particle paths

Above we have introduced the idea of a **fluid element** moving with a fluid, i.e., a small volume of fluid which at any time t and spatial position $\boldsymbol{r}$ has flow velocity $\boldsymbol{u}(t,\boldsymbol{r})$. We can now imagine the curve traced out by such a fluid element as it moves with a given flow. This curve is known as the **particle path**. It is the continuous set of loci a fluid element (labelled $\boldsymbol{q}$) has passed through on its motion with a fluid, during a given time interval. We obtain this curve as a solution to the ordinary differential differential equation of first-order

$$\left.\frac{\mathrm{d}\boldsymbol{r}(t,\boldsymbol{q})}{\mathrm{d}t}\right|_{\boldsymbol{q}} = \boldsymbol{u}[t,\boldsymbol{r}(t,\boldsymbol{q})]|_{\boldsymbol{q}} \ , \qquad (1.30)$$

subject to given initial conditions. Here $\boldsymbol{r}(t,\boldsymbol{q}) = [x(t),y(t),z(t)]^T|_{\boldsymbol{q}}$ denotes the position of the fluid element $\boldsymbol{q}$ at time t. In **Cartesian coordinates** this translates to the three ordinary differential equations of first-order (with $\boldsymbol{q}$ held fixed)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u_x[t,\boldsymbol{r}(t)] \ , \quad \frac{\mathrm{d}y}{\mathrm{d}t} = u_y[t,\boldsymbol{r}(t)] \ , \quad \frac{\mathrm{d}z}{\mathrm{d}t} = u_z[t,\boldsymbol{r}(t)] \ , \qquad (1.31)$$

subject to given initial conditions. In a **steady flow** the particle path is the *same* as a streamline.

Streaklines

Another experimental way to measure fluid flow is to introduce at a point a source of dye or smoke which is then continuously carried away from that point by the fluid flow. The resultant curve is called a **streakline**. An example is the line of smoke blown from a chimney. Once again, in a **steady flow** the streakline is the same as a streamline.

The streakline is the set of points at some fixed time $t =$ constant consisting of fluid elements that, *at some time t' in the past*, have all passed through some fixed point $\boldsymbol{c}$. Thus, the fluid element label $\boldsymbol{q}$ varies among all values such that $\boldsymbol{r}(t',\boldsymbol{q}) = \boldsymbol{c}$ for some time $t' < t$.

Example

Consider in **Cartesian coordinates** the 2-D, time-dependent, flow

$$\boldsymbol{u} = u_0(1,t,0)^T \ , \qquad u_0 = \text{constant} \ .$$

The **streamlines** are given by the equations

$$\mathrm{d}z = 0 \ , \quad \mathrm{d}y = t\mathrm{d}x \ , \qquad (1.32)$$

with $t =$ constant. The first equation gives $z = \mathrm{constant}$, so that the streamlines are parallel to the x–y plane; and the last equation can be integrated to give

$$y = tx + k \ , \qquad (1.33)$$

where k is the constant of integration. The streamlines are then *straight lines* with a gradient t; each value of k defines a different streamline.

For a **particle path**, consider the fluid element that has spatial position $(a, b, 0)^T$ at $t = 0$ (so that $u_y(t = 0) = 0$). Then the equations of motion are just

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u_0 \qquad \Rightarrow \qquad x = a + u_0 t \tag{1.34}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = u_0 t \qquad \Rightarrow \qquad y = b + \frac{1}{2}u_0 t^2 \tag{1.35}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = 0 \qquad \Rightarrow \qquad z = 0\ . \tag{1.36}$$

Eliminating t between solutions (1.34) and (1.35), we find

$$y = b + \frac{1}{2u_0}\,(x - a)^2\ . \tag{1.37}$$

Thus the particle paths are *parabolae*.

For a **streakline**, consider for example the one that passes through $(0, 0, 0)^T$. A fluid element position is specified by its initial spatial position, $(a, b, 0)^T$ (as above). It will pass through $(0, 0, 0)^T$ if

$$b + \frac{1}{2u_0}a^2 = 0\ , \tag{1.38}$$

from just putting $x = y = 0$ (and $z = 0$) in the particle path equation (1.37). From the particle path equations for x and y, (1.34) and (1.35), the fluid element at $(a, b, 0)^T$, which passed through $(0, 0, 0)^T$, did so at time t' given by

$$a = -\,u_0 t' \qquad b = -\,\frac{1}{2}\,u_0 t'^2\ . \tag{1.39}$$

The loci of all such points are found by substituting back into the particle path equations, so that

$$x = -\,u_0(t' - t)\ , \qquad y = -\,\frac{1}{2}\,u_0(t'^2 - t^2)\ , \qquad z = 0\ . \tag{1.40}$$

This gives $(x, y, z)^T$ for the streakline at time t parametrically in terms of t'. Finally, eliminating t' in (1.40) gives

$$y = \frac{1}{2}(2xt - \frac{x^2}{u_0}) \, . \tag{1.41}$$

Thus the streakline is also *parabolic*, but in the *opposite* sense to the particle paths.

1.6 Modelling fluids

When it comes to deducing the equations that govern fluid motion, there are two fundamentally different approaches: the **Eulerian description** and the **Lagrangian description**. The two viewpoints have been named in honour of the Swiss mathematician Leonhard Euler (1707–1783) and the French mathematician and mathematical physicist Joseph-Louis Lagrange (1736-1813), respectively.

1.6.1 Eulerian description

In the **Eulerian description** a **fixed reference frame** is employed relative to which a fluid is in motion. Time and spatial position in this reference frame, $\{t, \boldsymbol{r}\}$, are used as **independent variables**. The fluid variables such as mass density, pressure and flow velocity which describe the physical state of the fluid flow in question are **dependent variables** — as they are *functions* of the independent variables. Thus their derivatives are partial with respect to $\{t, \boldsymbol{r}\}$. For example, the flow velocity *at* a spatial position $\boldsymbol{r}$ and time t is given by $\boldsymbol{u}(t, \boldsymbol{r})$, and the corresponding acceleration at this position and time is then

$$\boldsymbol{a} = \left.\frac{\partial \boldsymbol{u}(t, \boldsymbol{r})}{\partial t}\right|_{\boldsymbol{r}} \, , \tag{1.42}$$

where the notation signifies that $\boldsymbol{r}$ is kept constant, i.e., the time derivative is for the *same* position.

1.6.2 Lagrangian description

In the **Lagrangian description** the fluid is described in terms of its constituent **fluid elements**. Different fluid elements have different “labels”, e.g., their spatial positions at a certain fixed time t, say $\boldsymbol{q}$. The **independent variables** are thus $\{t, \boldsymbol{q}\}$, and the particle position $\boldsymbol{r}(t, \boldsymbol{q})$ is a **dependent variable**. One can then ask about the rate of change in time in a reference frame *comoving* with the fluid element, and this then depends on time and particle label, i.e., which particular fluid element is being followed.

For example, if a fluid element has some velocity $\boldsymbol{u}(t, \boldsymbol{q})$, then the acceleration it feels will be

$$\boldsymbol{a} = \left. \frac{\partial \boldsymbol{u}(t, \boldsymbol{q})}{\partial t} \right|_{\boldsymbol{q}} , \tag{1.43}$$

where the notation signifies that $\boldsymbol{q}$ is kept constant, i.e., the time derivative is for the *same* fluid element.

1.6.3 Boundary conditions and initial conditions

Fluids need containers: they have boundaries. At a boundary one can find relationships between the fluid variables and the boundary, so-called **boundary conditions**. The boundary conditions will depend on the type of boundary (is it moving?), and the type of fluid (is the fluid viscous or inviscid?).

The types of fluid motion that we find depend on the **shape** and **symmetries** of the boundaries: water in a tea cup behaves differently from water in an ocean. It might also be important to consider the **scales** over which we are describing the fluid flow.

If there exist any symmetries in the boundaries, or the flow itself, then they are exceptionally important in determining the form of the fluid flow solutions.

- *Two-dimensional (2-D) flow*: The flow is independent of one spatial coordinate. For example, if in **Cartesian coordinates** $\{x, y, z\}$ it is independent of z, then $\boldsymbol{u}$, the flow velocity, takes the form

  $$\boldsymbol{u} = [u_x(t, x, y), u_y(t, x, y), 0]^T . \tag{1.44}$$

  A real configuration might produce approximately 2-D flow if the variation in the third dimension is sufficiently slow. For example, the flow over a wing might be approximately 2-D, except near the wing-tips.

- *Axisymmetric flow*: With suitable **cylindrical polar coordinates** $\{r, \varphi, z\}$ or **spherical polar coordinates** $\{r, \vartheta, \varphi\}$, all fluid variables are *independent of* φ (the azimuthal angle). There is still the possibility that the azimuthal flow velocity u_φ may be zero or non-zero (but still independent of φ).

The main purpose of **fluid dynamics** is to *predict* what kind of fluid motion we can expect to find for a given fluid configuration that experiences known internal and external forces. Thus, in order to use the **equations of motion** of **fluid dynamics** to predict the future development of a given (momentary) fluid configuration, one needs to know the **initial conditions** as well as the **boundary conditions** which describe the initial state of this configuration.

From a mathematical point of view, we can say that in the general case (i.e., when *no* simplifying assumptions are made) the **equations of motion** of **fluid dynamics** pose so-called **initial–boundary value problems**. In this course, for reasons of technical complexity, we will only marginally touch upon this general case.

1.6.4 Computational fluid dynamics

The **equations of motion** in **fluid dynamics** form a highly complicated **coupled system of non-linear partial differential equations** of first or second order. As expected from such a complicated system, they turn out to contain lots of interesting physical behaviour: **waves**, **shock wavess**, **turbulent flow**, and even **chaos**, as well as various simpler kinds of motions. As *no* general solution to this difficult equation system is available, and not even theorems proving existence, uniqueness and stability of solutions to the fully general equations of motions have been established, many situations in **fluid dynamics** can only be thoroughly studied by the use of **computer simulations**. Indeed, the modern field of **computational fluid dynamics (CFD)** is a vast and very active field of research in **physics** and **applied mathematics**.[4] At present we just restrict ourselves to the following remarks.

The distinction between different descriptions leads to different ways to solve problems of **fluid dynamics** on a computer. A computer can only handle a problem of finite size. In the **Eulerian description** the fluid domain is divided into a finite set of cells, each storing numbers for the fluid variables of a given flow. The **partial differential equations** of **fluid dynamics** are then solved numerically over this finite set of points.

In the **Lagrangian description** the fluid is divided into a finite set of fluid "elements" (or simulation particles), each of which has a stored position and velocity. The evolution of the fluid is simulated by solving the equations of motion for all of the simulation particles. If the fluid variables are required at any particular point in space, then any simulation particles in the vicinity of that point are found and then the corresponding quantities averaged.

[4]Take, e.g., a look at the websites listed at the beginning of these lecture notes.

Chapter 2
Mathematical techniques

2.1 Overview

On the mathematical side, our study of **fluid dynamics** is in terms of scalar fields (e.g., the fluid mass density and fluid pressure) and vector fields (e.g., the fluid flow velocity) defined on Euclidian space $\mathbb{R}^3$, and the relationships between them. Since these relationships are mostly *differential* in nature, we make extensive use of **vector calculus**. This chapter provides a brief review of the mathematical techniques we will employ in this study.

First we will review **vector calculus** for different coordinate systems on $\mathbb{R}^3$. Then we will briefly discuss two fundamental **integral theorems**. Both these topics are introduced to you in more detail in MAS204 Calculus III. Next we address the issue of **matrix transformations**, which are transformations that describe a *linear* change of variables. And, finally, following the introduction of **Dirac's delta function**, we will discuss the general expansion of a square integrable real-valued function $f(x)$ over a given interval in terms of **complete sets of orthonormal functions**, the most familiar one being perhaps **Fourier series expansions**.

2.2 Vector calculus in Cartesian coordinates

Let us remind ourselves of the vector analytical differential operators in a right-handed oriented **Cartesian coordinate basis** of $\mathbb{R}^3$, $\{\, \boldsymbol{e}_x,\ \boldsymbol{e}_y,\ \boldsymbol{e}_z \,\}$, with coordinates $\{x, y, z\}$, where

$$1 = |\, \boldsymbol{e}_x \,| = |\, \boldsymbol{e}_y \,| = |\, \boldsymbol{e}_z \,| \ .$$

If on a domain $D \subset \mathbb{R}^3$ we have ϕ as a differentiable scalar-valued function of position $\boldsymbol{r} = (x, y, z)^T$, and $\boldsymbol{A} = (A_x, A_y, A_z)^T$ as a differentiable vector-valued function of $\boldsymbol{r}$, then the vector analytical differential operators assume the explicit forms

Gradient operator:

$$\boldsymbol{\nabla}\phi \ = \ \frac{\partial\phi}{\partial x}\,\boldsymbol{e}_x + \frac{\partial\phi}{\partial y}\,\boldsymbol{e}_y + \frac{\partial\phi}{\partial z}\,\boldsymbol{e}_z \ . \qquad (2.1)$$

Divergence operator:

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} \;=\; \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \,. \tag{2.2}$$

Curl operator:

$$\begin{aligned} \boldsymbol{\nabla} \times \boldsymbol{A} \;=\; & \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right] \boldsymbol{e}_x + \left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right] \boldsymbol{e}_y \\ & + \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right] \boldsymbol{e}_z \,. \end{aligned} \tag{2.3}$$

Laplace operator:

$$(\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\phi \;=\; \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \,. \tag{2.4}$$

2.3 Vector calculus in orthogonal curvilinear coordinates

Assume on $\mathbb{R}^3$ given a right-handed oriented **orthogonal coordinate basis** $\{\, \boldsymbol{e}_1,\, \boldsymbol{e}_2,\, \boldsymbol{e}_3 \,\}$, with coordinates $\{x_1, x_2, x_3\}$, where orthogonal means that $\boldsymbol{e}_i \cdot \boldsymbol{e}_j = 0$ for $i \neq j$, and $i, j = 1, 2, 3$. From this a **normalised orthogonal basis**[1] $\{\, \hat{\boldsymbol{e}}_1,\, \hat{\boldsymbol{e}}_2,\, \hat{\boldsymbol{e}}_3 \,\}$ can be obtained, where normalised means that

$$1 = \hat{\boldsymbol{e}}_1 \cdot \hat{\boldsymbol{e}}_1 = \hat{\boldsymbol{e}}_2 \cdot \hat{\boldsymbol{e}}_2 = \hat{\boldsymbol{e}}_3 \cdot \hat{\boldsymbol{e}}_3 \,.$$

Introducing the quantities

$$g_{11} := \boldsymbol{e}_1 \cdot \boldsymbol{e}_1 \,, \qquad g_{22} := \boldsymbol{e}_2 \cdot \boldsymbol{e}_2 \,, \qquad g_{33} := \boldsymbol{e}_3 \cdot \boldsymbol{e}_3 \,,$$

the two sets of basis vectors are related by the definitions

$$\hat{\boldsymbol{e}}_1 := \frac{1}{\sqrt{g_{11}}}\, \boldsymbol{e}_1 \,, \qquad \hat{\boldsymbol{e}}_2 := \frac{1}{\sqrt{g_{22}}}\, \boldsymbol{e}_2 \,, \qquad \hat{\boldsymbol{e}}_3 := \frac{1}{\sqrt{g_{33}}}\, \boldsymbol{e}_3 \,.$$

Let, on a domain $D \subset \mathbb{R}^3$, $\phi = \phi(x_1, x_2, x_3)$ be a differentiable scalar-valued function, and $\boldsymbol{A} = \boldsymbol{A}(x_1, x_2, x_3)$ a differentiable vector-valued function. We have

$$\boldsymbol{A} = {}^{c}\!A_1\, \boldsymbol{e}_1 + {}^{c}\!A_2\, \boldsymbol{e}_2 + {}^{c}\!A_3\, \boldsymbol{e}_3 = A_1\, \hat{\boldsymbol{e}}_1 + A_2\, \hat{\boldsymbol{e}}_2 + A_3\, \hat{\boldsymbol{e}}_3 \,,$$

where

$$A_1 = \sqrt{g_{11}}\, {}^{c}\!A_1 \,, \qquad A_2 = \sqrt{g_{22}}\, {}^{c}\!A_2 \,, \qquad A_3 = \sqrt{g_{33}}\, {}^{c}\!A_3 \,.$$

One can show that with respect to the normalised orthogonal basis $\{\, \hat{\boldsymbol{e}}_1,\, \hat{\boldsymbol{e}}_2,\, \hat{\boldsymbol{e}}_3 \,\}$, the vector analytical differential operators are given by

[1] This is, in general, a *non-coordinate basis*, i.e., it *cannot* be generated from a transformation of an (orthogonal) coordinate basis $\{\, \boldsymbol{e}_1,\, \boldsymbol{e}_2,\, \boldsymbol{e}_3 \,\}$ with a Jacobian matrix of a specific coordinate transformation.

Gradient operator:

$$\nabla\phi = \frac{1}{\sqrt{g_{11}}}\frac{\partial\phi}{\partial x_1}\,\hat{e}_1 + \frac{1}{\sqrt{g_{22}}}\frac{\partial\phi}{\partial x_2}\,\hat{e}_2 + \frac{1}{\sqrt{g_{33}}}\frac{\partial\phi}{\partial x_3}\,\hat{e}_3 \qquad (2.5)$$

Divergence operator:

$$\nabla\cdot A = \frac{1}{\sqrt{g_{11}g_{22}g_{33}}} \times\left[\frac{\partial}{\partial x_1}(\sqrt{g_{22}g_{33}}\,A_1) + \frac{\partial}{\partial x_2}(\sqrt{g_{33}g_{11}}\,A_2) + \frac{\partial}{\partial x_3}(\sqrt{g_{11}g_{22}}\,A_3)\right] \qquad (2.6)$$

Curl operator:

$$\nabla\times A = \frac{1}{\sqrt{g_{11}g_{22}g_{33}}}\left[\left(\frac{\partial(\sqrt{g_{33}}\,A_3)}{\partial x_2} - \frac{\partial(\sqrt{g_{22}}\,A_2)}{\partial x_3}\right)\sqrt{g_{11}}\,\hat{e}_1 + \left(\frac{\partial(\sqrt{g_{11}}\,A_1)}{\partial x_3} - \frac{\partial(\sqrt{g_{33}}\,A_3)}{\partial x_1}\right)\sqrt{g_{22}}\,\hat{e}_2 + \left(\frac{\partial(\sqrt{g_{22}}\,A_2)}{\partial x_1} - \frac{\partial(\sqrt{g_{11}}\,A_1)}{\partial x_2}\right)\sqrt{g_{33}}\,\hat{e}_3\right] \qquad (2.7)$$

Laplace operator:

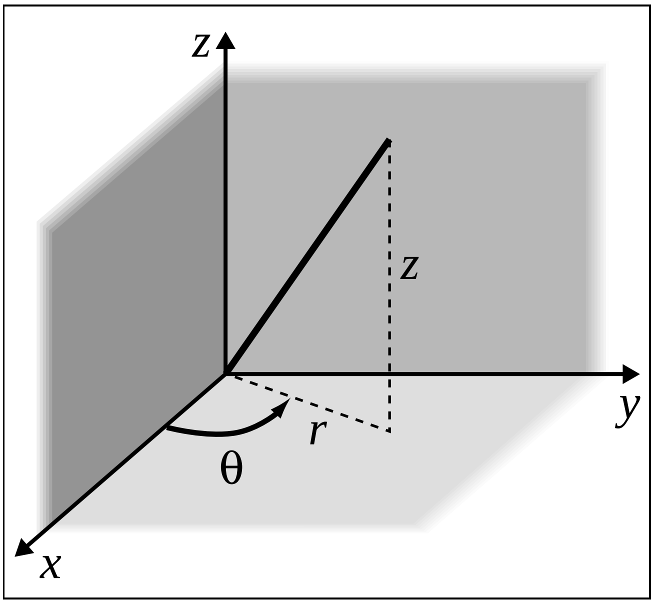
$$(\nabla\cdot\nabla)\phi = \frac{1}{\sqrt{g_{11}g_{22}g_{33}}} \times\left[\frac{\partial}{\partial x_1}\left(\sqrt{\frac{g_{22}g_{33}}{g_{11}}}\,\frac{\partial\phi}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\sqrt{\frac{g_{33}g_{11}}{g_{22}}}\,\frac{\partial\phi}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\sqrt{\frac{g_{11}g_{22}}{g_{33}}}\,\frac{\partial\phi}{\partial x_3}\right)\right] , \qquad (2.8)$$

respectively. The last result follows from (2.6) when $A_1 = \frac{1}{\sqrt{g_{11}}}\,\partial\phi/\partial x_1$, $A_2 = \frac{1}{\sqrt{g_{22}}}\,\partial\phi/\partial x_2$ and $A_3 = \frac{1}{\sqrt{g_{33}}}\,\partial\phi/\partial x_3$ are used. We are interested in the explicit form these vector analytical differential operators assume for two frequently employed kinds of **orthogonal curvilinear coordinates**.

2.3.1 Cylindrical polar coordinates

Cartesian coordinates $\{x, y, z\}$ on $\mathbb{R}^3$ are related to **cylindrical polar coordinates** $\{r, \varphi, z\}$ according to

$$x = r\cos\varphi\ , \quad y = r\sin\varphi\ , \quad z = z\ , \qquad (2.9)$$



where the coordinates $\{r, \varphi, z\}$ vary in the intervals $r \geq 0$, $0 \leq \varphi \leq 2\pi$ and $-\infty < z < +\infty$. The coordinate r gives the magnitude of the position vector $\boldsymbol{r}$ projected onto a plane $z =$ const, while z is the magnitude of $\boldsymbol{r}$ projected onto the z–axis. The coordinate φ is the azimuthal angle subtended by the projection of $\boldsymbol{r}$ onto a plane $z =$ const and the positive (x, z)–half plane, measured anti-clockwise. *Note* that sometimes θ is used to denote the azimuthal angle, instead of φ.

The right-handed oriented coordinate basis $\{\,\boldsymbol{e}_r,\,\boldsymbol{e}_\varphi,\,\boldsymbol{e}_z\,\}$ is given in terms of the Cartesian coordinate basis $\{\,\boldsymbol{e}_x,\,\boldsymbol{e}_y,\,\boldsymbol{e}_z\,\}$ by

$$\begin{aligned} \boldsymbol{e}_r &= \cos\varphi\,\boldsymbol{e}_x + \sin\varphi\,\boldsymbol{e}_y \\ \boldsymbol{e}_\varphi &= -\,r\sin\varphi\,\boldsymbol{e}_x + r\cos\varphi\,\boldsymbol{e}_y \quad . \\ \boldsymbol{e}_z &= \boldsymbol{e}_z \end{aligned} \tag{2.10}$$

For cylindrical polar coordinates we thus have

$$\sqrt{g_{11}} = \sqrt{g_{rr}} = 1\;, \qquad \sqrt{g_{22}} = \sqrt{g_{\varphi\varphi}} = r\;, \qquad \sqrt{g_{33}} = \sqrt{g_{zz}} = 1\;,$$

so that we define a normalised orthogonal basis $\{\,\hat{\boldsymbol{e}}_r,\,\hat{\boldsymbol{e}}_\varphi,\,\hat{\boldsymbol{e}}_z\,\}$ by

$$\hat{\boldsymbol{e}}_r := \boldsymbol{e}_r\;, \qquad \hat{\boldsymbol{e}}_\varphi := \frac{1}{r}\,\boldsymbol{e}_\varphi\;, \qquad \hat{\boldsymbol{e}}_z := \boldsymbol{e}_z\;. \tag{2.11}$$

Then we obtain from (2.5)–(2.8), respectively,

Gradient operator:

$$\boldsymbol{\nabla}\phi = \frac{\partial\phi}{\partial r}\,\hat{\boldsymbol{e}}_r + \frac{1}{r}\,\frac{\partial\phi}{\partial\varphi}\,\hat{\boldsymbol{e}}_\varphi + \frac{\partial\phi}{\partial z}\,\hat{\boldsymbol{e}}_z \tag{2.12}$$

Divergence operator:

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} = \frac{1}{r}\,\frac{\partial}{\partial r}(r\,A_r) + \frac{1}{r}\,\frac{\partial A_\varphi}{\partial\varphi} + \frac{\partial A_z}{\partial z} \tag{2.13}$$

Curl operator:

$$\nabla \times \boldsymbol{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right)\hat{\boldsymbol{e}}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\boldsymbol{e}}_\varphi + \frac{1}{r}\left(\frac{\partial}{\partial r}(r\,A_\varphi) - \frac{\partial A_r}{\partial \varphi}\right)\hat{\boldsymbol{e}}_z \qquad (2.14)$$

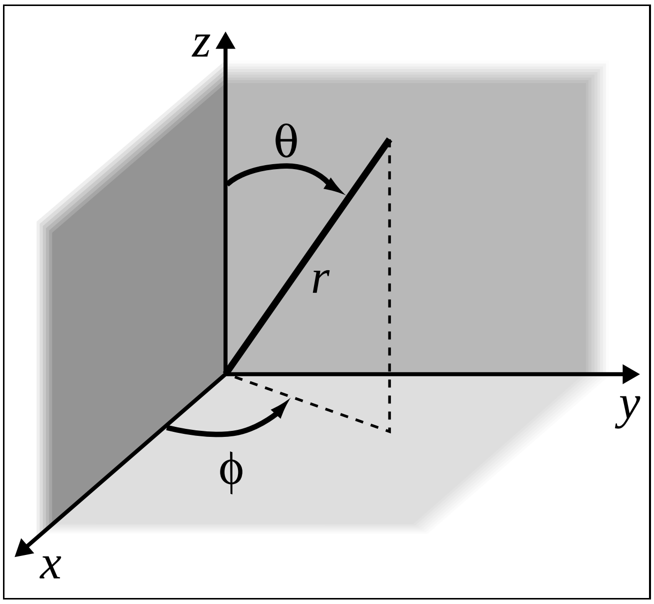
Laplace operator:

$$(\nabla \cdot \nabla)\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \varphi^2} + \frac{\partial^2 \phi}{\partial z^2}\,. \qquad (2.15)$$

2.3.2 Spherical polar coordinates

Cartesian coordinates $\{x, y, z\}$ on $\mathbb{R}^3$ are related to **spherical polar coordinates** $\{r, \vartheta, \varphi\}$ according to

$$x = r\sin\vartheta\cos\varphi\,, \quad y = r\sin\vartheta\sin\varphi\,, \quad z = r\cos\vartheta\,, \qquad (2.16)$$



where the coordinates $\{r, \vartheta, \varphi\}$ vary in the intervals $r \geq 0$, $0 \leq \vartheta \leq \pi$, and $0 \leq \varphi \leq 2\pi$. The coordinate r gives the magnitude of the position vector $\boldsymbol{r}$, while the coordinate ϑ denotes the angle between $\boldsymbol{r}$ and the z–axis. The coordinate φ is the azimuthal angle subtended by the projection of $\boldsymbol{r}$ onto the (x, y)–plane and the positive (x, z)–half plane, measured anti-clockwise.

The right-handed oriented coordinate basis $\{\,\boldsymbol{e}_r,\,\boldsymbol{e}_\vartheta,\,\boldsymbol{e}_\varphi\,\}$ is given in terms of the Cartesian coordinate basis $\{\,\boldsymbol{e}_x,\,\boldsymbol{e}_y,\,\boldsymbol{e}_z\,\}$ by

$$\begin{aligned} \boldsymbol{e}_r &= \sin\vartheta\cos\varphi\,\boldsymbol{e}_x + \sin\vartheta\sin\varphi\,\boldsymbol{e}_y + \cos\vartheta\,\boldsymbol{e}_z \\ \boldsymbol{e}_\vartheta &= r\cos\vartheta\cos\varphi\,\boldsymbol{e}_x + r\cos\vartheta\sin\varphi\,\boldsymbol{e}_y - r\sin\vartheta\,\boldsymbol{e}_z \quad . \\ \boldsymbol{e}_\varphi &= -\,r\sin\vartheta\sin\varphi\,\boldsymbol{e}_x + r\sin\vartheta\cos\varphi\,\boldsymbol{e}_y \end{aligned} \qquad (2.17)$$

For spherical polar coordinates we thus have

$$\sqrt{g_{11}} = \sqrt{g_{rr}} = 1 \ , \qquad \sqrt{g_{22}} = \sqrt{g_{\vartheta\vartheta}} = r \ , \qquad \sqrt{g_{33}} = \sqrt{g_{\varphi\varphi}} = r\,\sin\vartheta \ ,$$

so that we define a normalised orthogonal basis $\{\,\hat{\boldsymbol{e}}_r,\,\hat{\boldsymbol{e}}_\vartheta,\,\hat{\boldsymbol{e}}_\varphi\,\}$ by

$$\hat{\boldsymbol{e}}_r := \boldsymbol{e}_r \ , \qquad \hat{\boldsymbol{e}}_\vartheta := \frac{1}{r}\,\boldsymbol{e}_\vartheta \ , \qquad \hat{\boldsymbol{e}}_\varphi := \frac{1}{r\sin\varphi}\,\boldsymbol{e}_\varphi \ . \tag{2.18}$$

Then we obtain from (2.5)–(2.8), respectively,

Gradient operator:

$$\boldsymbol{\nabla}\phi \;=\; \frac{\partial\phi}{\partial r}\,\hat{\boldsymbol{e}}_r + \frac{1}{r}\,\frac{\partial\phi}{\partial\vartheta}\,\hat{\boldsymbol{e}}_\vartheta + \frac{1}{r\sin\vartheta}\,\frac{\partial\phi}{\partial\varphi}\,\hat{\boldsymbol{e}}_\varphi \tag{2.19}$$

Divergence operator:

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} \;=\; \frac{1}{r^2}\,\frac{\partial}{\partial r}(r^2\,A_r) + \frac{1}{r\sin\vartheta}\,\frac{\partial}{\partial\vartheta}(\sin\vartheta\,A_\vartheta) + \frac{1}{r\sin\vartheta}\,\frac{\partial A_\varphi}{\partial\varphi} \tag{2.20}$$

Curl operator:

$$\begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{A} \;=\;& \frac{1}{r\sin\vartheta}\left(\frac{\partial}{\partial\vartheta}(\sin\vartheta\,A_\varphi) - \frac{\partial A_\vartheta}{\partial\varphi}\right)\hat{\boldsymbol{e}}_r \\ &+ \frac{1}{r}\left(\frac{1}{\sin\vartheta}\,\frac{\partial A_r}{\partial\varphi} - \frac{\partial}{\partial r}(r\,A_\varphi)\right)\hat{\boldsymbol{e}}_\vartheta + \frac{1}{r}\left(\frac{\partial}{\partial r}(r\,A_\vartheta) - \frac{\partial A_r}{\partial\vartheta}\right)\hat{\boldsymbol{e}}_\varphi\end{aligned} \tag{2.21}$$

Laplace operator:

$$\begin{aligned}(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\phi \;=\;& \frac{1}{r^2}\,\frac{\partial}{\partial r}\left(r^2\,\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\,\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\,\frac{\partial\phi}{\partial\vartheta}\right) \\ &+ \frac{1}{r^2\sin^2\vartheta}\,\frac{\partial^2\phi}{\partial\varphi^2} \ .\end{aligned} \tag{2.22}$$

2.4 Vector analytical identities

Certain vector analytical identities will be of use in this course, which do hold *independent* of the choice of coordinate system we pick on $\mathbb{R}^3$. Let, on a domain $D \subset \mathbb{R}^3$, $\phi = \phi(\boldsymbol{r})$ and $\psi = \psi(\boldsymbol{r})$ be differentiable scalar-valued functions, and $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{r})$ and $\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{r})$ be differentiable vector-valued functions. Then

$$\boldsymbol{\nabla}\cdot(\phi\,\boldsymbol{A}) \;\equiv\; \phi\,(\boldsymbol{\nabla}\cdot\boldsymbol{A}) + \boldsymbol{A}\cdot(\boldsymbol{\nabla}\phi) \tag{2.23}$$

$$\boldsymbol{\nabla}\times(\phi\,\boldsymbol{A}) \;\equiv\; \phi\,(\boldsymbol{\nabla}\times\boldsymbol{A}) - \boldsymbol{A}\times(\boldsymbol{\nabla}\phi) \tag{2.24}$$

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\phi) \;\equiv\; \boldsymbol{0} \tag{2.25}$$

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A}) \;\equiv\; 0 \tag{2.26}$$

$$\boldsymbol{\nabla}(\phi\,\psi) \;\equiv\; \psi\,\boldsymbol{\nabla}\phi + \phi\,\boldsymbol{\nabla}\psi \tag{2.27}$$

$$\begin{aligned} \boldsymbol{\nabla}\cdot(\boldsymbol{A}\times\boldsymbol{B}) &\equiv \boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A}) - \boldsymbol{A}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B}) \quad (2.28)\\ \boldsymbol{\nabla}\times(\boldsymbol{A}\times\boldsymbol{B}) &\equiv (\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{A} - (\boldsymbol{A}\cdot\boldsymbol{\nabla})\boldsymbol{B} \quad (2.29)\\ &\quad + \boldsymbol{A}\,(\boldsymbol{\nabla}\cdot\boldsymbol{B}) - \boldsymbol{B}\,(\boldsymbol{\nabla}\cdot\boldsymbol{A})\\ \boldsymbol{\nabla}(\boldsymbol{A}\cdot\boldsymbol{B}) &\equiv \boldsymbol{A}\times(\boldsymbol{\nabla}\times\boldsymbol{B}) + \boldsymbol{B}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) \quad (2.30)\\ &\quad + (\boldsymbol{A}\cdot\boldsymbol{\nabla})\boldsymbol{B} + (\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{A}\,. \end{aligned}$$

Note that with respect to a **Cartesian coordinate basis** (only) the additional identity

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) \equiv \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A}) - (\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\boldsymbol{A} \quad (2.31)$$

holds.

2.5 Integral theorems

2.5.1 Gauß' integral theorem

Consider in Euclidian space $\mathbb{R}^3$ a simply-connected **volume** G that is bounded by a **closed surface** ∂G, and a differentiable **vector field** $\boldsymbol{A} = \boldsymbol{A}(t,\boldsymbol{r})$ defined everywhere throughout the region that contains G. We assume that ∂G is fixed in time. Then we have[2]

$$\iiint_G \boldsymbol{\nabla}\cdot\boldsymbol{A}\,\mathrm{d}V = \iint_{\partial G} \boldsymbol{A}\cdot\boldsymbol{n}\,\mathrm{d}A\,, \quad (2.32)$$

where $\boldsymbol{n}$ is the outward-pointing **unit normal** to ∂G. This theorem is named after the German mathematician and astronomer Carl Friedrich Gauß (1777–1855) and dates back to the early nineteenth century.

2.5.2 Stokes' integral theorem

Consider in Euclidian space $\mathbb{R}^3$ a simply-connected **surface** S that is bounded by an oriented **closed curve** ∂S, and a differentiable **vector field** $\boldsymbol{A} = \boldsymbol{A}(t,\boldsymbol{r})$ defined everywhere throughout the region that contains S. We assume that ∂S is fixed in time. Then we have

$$\iint_S (\boldsymbol{\nabla}\times\boldsymbol{A})\cdot\boldsymbol{n}\,\mathrm{d}A = \oint_{\partial S} \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{s}\,, \quad (2.33)$$

where $\boldsymbol{n}$ is the **unit normal** to S, directed in relation to the orientation on ∂S according to a right-hand convention. This theorem is named after the Irish mathematician and physicist George Gabriel Stokes (1819–1903) who established a proof during the mid-nineteenth century.

[2] The surface integral sign on the right-hand side should really be a two-dimensional analog of the closed-loop line integral sign $\oint$, to emphasise that the integration is over a *closed* surface. However, it appears that LaTeX2ε does not hold such a symbol available (or at least we have not discovered it yet). Any help would be appreciated.

2.6 Cartesian index notation

We can prove the above vector identities, and others, without resorting to integral theorems, using expressions for the various vector operators in any particular coordinate system. In order to avoid long cumbersome calculations, we use the **index notation** and the **summation convention**.

With respect to a Cartesian coordinate basis, **position vectors** are written as x_i, $i = 1, 2, 3$, and **general vectors** A_i, $i = 1, 2, 3$. **Partial differentiation** is then written as $\partial/\partial x_i$ or ∂_i.

In order to write **scalar** and **vector products** one introduces the **Kronecker symbol** (named after the German mathematician Leopold Kronecker, 1823–1891)

$$\delta_{ij} := \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} , \tag{2.34}$$

and the **alternating symbol**

$$\epsilon_{ijk} := \begin{cases} +1 & \text{cyclic (or even) permutations of (1,2,3)} \\ -1 & \text{anticyclic (or odd) permutations of (1,2,3)} \\ 0 & \text{otherwise} \end{cases} . \tag{2.35}$$

The **summation convention** is that any repeated indices are summed over $1, 2, 3$, and the summation sign is omitted. Summation indices are sometimes called "**dummy indices**", since they can be changed to another value (i.e., symbol) without affecting the outcome: e.g., (trivially) $A_i B_i = A_j B_j$. Using the summation convention leads to the following useful **identities**:

$$\delta_{ii} = 3 \tag{2.36}$$

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \tag{2.37}$$

$$\epsilon_{ijk}\epsilon_{ijm} = 2\delta_{km} \tag{2.38}$$

$$\epsilon_{ijk}\epsilon_{ijk} = 6 . \tag{2.39}$$

The following shows familiar **differential operators** expressed by using the summation convention:

$$(\nabla\phi)_i = \partial_i\phi \tag{2.40}$$

$$\nabla \cdot \boldsymbol{A} = \partial_i A_i \tag{2.41}$$

$$(\boldsymbol{A} \times \boldsymbol{B})_i = \epsilon_{ijk} A_j B_k \tag{2.42}$$

$$\boldsymbol{A} \cdot \boldsymbol{B} = A_i B_i \tag{2.43}$$

$$(\nabla \times \boldsymbol{A})_i = \epsilon_{ijk}\partial_j A_k . \tag{2.44}$$

We also note a useful property: If T_{ij} is **symmetric** (i.e., $T_{ij} = T_{ji}$), then

$$\epsilon_{ijk} T_{jk} \equiv 0 , \tag{2.45}$$

from the anti-symmetric properties of ϵ_{ijk}. Obviously it follows that

$$\epsilon_{ijk} A_j A_k \equiv 0 , \tag{2.46}$$

which is none other than the identity $\boldsymbol{A} \times \boldsymbol{A} \equiv \boldsymbol{0}$.

And now, a worked example:

$$\begin{aligned}
(\boldsymbol{\nabla} \times (\boldsymbol{A} \times \boldsymbol{B}))_i &= \epsilon_{ijk}\partial_j(\boldsymbol{A} \times \boldsymbol{B})_k \\
&= \epsilon_{ijk}\partial_j(\epsilon_{klm}A_l B_m) \\
&= \epsilon_{kij}\epsilon_{klm}\partial_j(A_l B_m) \\
&= \delta_{il}\delta_{jm}\partial_j(A_l B_m) - \delta_{lm}\delta_{jl}\partial_j(A_l B_m) \\
&= \partial_j(A_i B_j) - \partial_j(A_j B_i) \\
&= A_i\partial_j B_j + B_j\partial_j A_i - B_i\partial_j A_j - A_j\partial_j B_i \\
&= (\boldsymbol{\nabla} \cdot \boldsymbol{B})(\boldsymbol{A})_i + (\boldsymbol{B} \cdot \boldsymbol{\nabla})(\boldsymbol{A})_i \\
&\quad - (\boldsymbol{\nabla} \cdot \boldsymbol{A})(\boldsymbol{B})_i - (\boldsymbol{A} \cdot \boldsymbol{\nabla})(\boldsymbol{B})_i \,.
\end{aligned} \tag{2.47}$$

Note, how we use the properties of the alternating symbol to “rotate” its indices (e.g., $\epsilon_{ijk} = \epsilon_{kij}$), and the properties of the Kronecker symbol to “remove” dummy indices (e.g., $\delta_{ij}B_j = B_i$).

2.7 Matrices and linear transformations

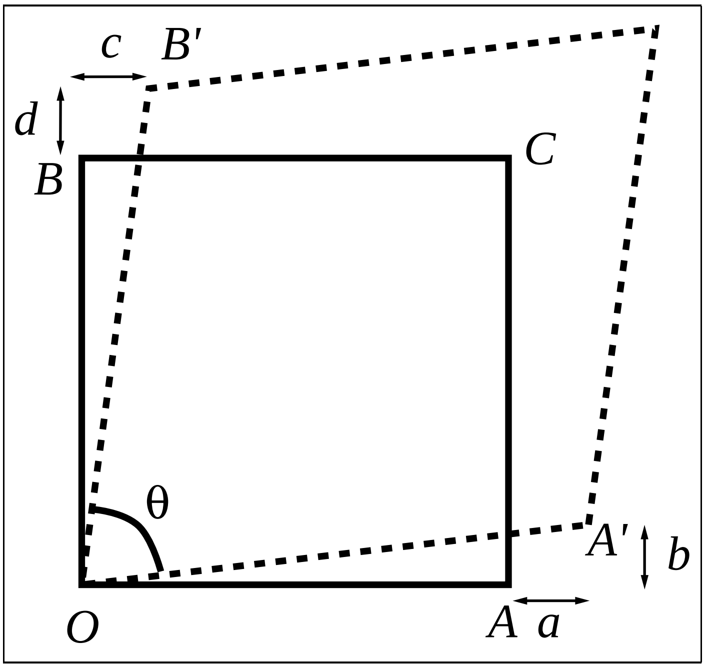
Matrices can be regarded as corresponding to **linear geometrical transformations** that act on vectors and transform them into other vectors. This is relevant to the study of **fluid dynamics** because a **fluid line element** can be transformed into a fluid line element of a different magnitude and pointing in another direction due to **differential variations** in the **flow velocity field** of the fluid. Also, there is a relationship between the vector **normal** to any particular surface element in the fluid, and the vector representing the **force** on that surface element. These two vectors are *not*, in general, parallel to each other, so the relationship must involve a linear transformation described by a matrix.

Rather than work in a general way, we will present results for 2-D matrix transformations. The corresponding 3-D results can be derived in a similar fashion.

Consider what happens when a unit square with vertices in Cartesian coordinates at $(0,0)^T$, $(1,0)^T$, $(0,1)^T$ and $(1,1)^T$ (labelled O, A, B and C, respectively) is transformed by a matrix defined by

$$\mathsf{1} + \mathsf{M} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1+a & b \\ c & 1+d \end{pmatrix} , \tag{2.48}$$

which, as the constants a, b, c and d shall be assumed to be *very much smaller than unity*, deviates only slightly from the unit matrix. In other words, this matrix will produce only *small* changes.



The vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$, originally orthogonal, are transformed into new vectors

$$\overrightarrow{OA'} = (1+a)\,\boldsymbol{e}_x + c\,\boldsymbol{e}_y \tag{2.49}$$

$$\overrightarrow{OB'} = b\,\boldsymbol{e}_x + (1+d)\,\boldsymbol{e}_y \;. \tag{2.50}$$

The **angle** φ between the new $\overrightarrow{OA'}$ and $\overrightarrow{OB'}$ is

$$\cos\varphi = \frac{\overrightarrow{OA'}\cdot\overrightarrow{OB'}}{|\,\overrightarrow{OA'}\,|\,|\,\overrightarrow{OB'}\,|} \approx (b+c)\;, \tag{2.51}$$

where only terms of *first-order smallness* in a, b, c and d have been retained in the final expression. Similarly, to *first-order smallness* in a, b, c and d, the **area** A spanned by the new $\overrightarrow{OA'}$ and $\overrightarrow{OB'}$ is

$$A = |\,\overrightarrow{OA'}\,|\,|\,\overrightarrow{OB'}\,|\,\sin\varphi \approx 1 + (a+d)\;. \tag{2.52}$$

Generalizing from this simple example, we can make the following statements about the action of the matrix M:

- An **anti-symmetric** M, with $a = d = 0$ and $b = -c$, leaves both the angle and the area between two vectors unchanged. This corresponds to *rigid rotation*.
- The *change in area* of a unit square equals the **trace** of M, i.e., the sum of its diagonal elements, $a + d$.

- A **symmetric–trace-free** M, with $b = c$ and $d = -a$, leaves the area between two vectors unchanged but changes the angle. This corresponds to *shearing*.

We can rewrite the matrix M representing the *change* in the vectors spanning a unit square as an *irreducible* sum of three matrices, each with a different geometrical effect:

$$\mathsf{M} = \frac{1}{2}(\mathrm{Tr}\,\mathsf{M})\,\mathsf{1} + \left[\frac{1}{2}(\mathsf{M}+\mathsf{M}^T) - \frac{1}{2}(\mathrm{Tr}\,\mathsf{M})\,\mathsf{1}\right] + \frac{1}{2}(\mathsf{M}-\mathsf{M}^T)$$

$\Rightarrow$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \underbrace{\frac{1}{2}(a+d)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{Expansion}} + \underbrace{\frac{1}{2}\begin{pmatrix} (a-d) & (b+c) \\ (b+c) & -(a-d) \end{pmatrix}}_{\text{Shear}} + \underbrace{\frac{1}{2}\begin{pmatrix} 0 & (b-c) \\ -(b-c) & 0 \end{pmatrix}}_{\text{Rotation}} . \qquad (2.53)$$

The first results just in a multiplication by a scalar, i.e, an overall **expansion** (or **contraction**). The second represents a trace-free distortion, i.e., a change of shape without change of area, or **shear**. The third is anti-symmetric, i.e., it generates a rigid **rotation**. Thus any linear transformation involving only *small* (i.e., infinitesimal) changes of a geometrical object can be thought of as the sum of an expansion, a shear, and a rotation of that object.

A further point: **shear** is equivalent to **anisotropic expansion/contraction**, typically along orthogonal directions *not* aligned with the coordinate directions. To demonstrate this, note that the first two matrix components (expansion and shear) are symmetric, but the second (or sum of first and second) is not necessarily diagonal. But it is possible to choose particular vectors (eigenvectors) such that

$$\mathsf{S}\,\boldsymbol{v} = \lambda \boldsymbol{v} \, . \qquad (2.54)$$

This implies that the matrix S acts on vectors parallel to $\boldsymbol{v}$ by just multiplying them by a scalar, i.e., it does not change their orientations.

As an example consider the shear matrix

$$\mathsf{S} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \qquad (2.55)$$

The eigenvalue equation becomes

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \, , \qquad (2.56)$$

so that the eigenvalues of S are $\lambda = \pm 1$, and the eigenvectors are $(1/\sqrt{2})(1,1)^T$ and $(1/\sqrt{2})(1,-1)^T$. In the present case, the shear is equivalent to a pure expansion/compression along axes at 45° to the original Cartesian directions.

To conclude this section, let us state in view of later applications that in 3-D a square matrix M is decomposed into its irreducible parts according to

$$\mathsf{M} = \frac{1}{3}\,(\mathrm{Tr}\,\mathsf{M})\,\mathsf{1} + \left[\,\frac{1}{2}\,(\mathsf{M}+\mathsf{M}^T) - \frac{1}{3}\,(\mathrm{Tr}\,\mathsf{M})\,\mathsf{1}\,\right] + \frac{1}{2}\,(\mathsf{M}-\mathsf{M}^T)\;.$$

2.8 Dirac's delta function

Let us assume that $f(\boldsymbol{r})$ is a continuously differentiable real-valued function on a region $G \subset \mathbb{R}^3$. Then, by definition, **Dirac's delta function** (named after the English physicist Paul Adrien Maurice Dirac, 1902–1984) has the property

$$\int\!\!\int\!\!\int_G f(\boldsymbol{r}')\,\delta(\boldsymbol{r}-\boldsymbol{r}')\,\mathrm{d}V' = \begin{cases} f(\boldsymbol{r}) & \text{for } \boldsymbol{r}=\boldsymbol{r}' \in G \\ 0 & \text{for } \boldsymbol{r}=\boldsymbol{r}' \notin G \end{cases} \;. \tag{2.57}$$

With the special choice $f(\boldsymbol{r}) = 1$, this yields

$$\int\!\!\int\!\!\int_G \delta(\boldsymbol{r}-\boldsymbol{r}')\,\mathrm{d}V' = \begin{cases} 1 & \text{for } \boldsymbol{r}=\boldsymbol{r}' \in G \\ 0 & \text{for } \boldsymbol{r}=\boldsymbol{r}' \notin G \end{cases} \;. \tag{2.58}$$

Assuming compact support for $f(\boldsymbol{r})$ on G (i.e., sufficiently rapid fall-off behaviour as $r \to \infty$), a gradient of **Dirac's delta function** can be interpreted according to

$$\int\!\!\int\!\!\int_G f(\boldsymbol{r}')\,\boldsymbol{\nabla}_{\boldsymbol{r}'}\delta(\boldsymbol{r}-\boldsymbol{r}')\,\mathrm{d}V' = -\,\boldsymbol{\nabla}_{\boldsymbol{r}'}f(\boldsymbol{r}')\,\big|_{\boldsymbol{r}'=\boldsymbol{r}} \;; \tag{2.59}$$

$\boldsymbol{\nabla}_{\boldsymbol{r}'}$ here denotes the gradient with respect to $\boldsymbol{r}'$. Note that $\delta(\boldsymbol{r}-\boldsymbol{r}') = \delta(\boldsymbol{r}'-\boldsymbol{r})$ applies, i.e., **Dirac's delta function** is symmetric.

2.9 Orthonormal function expansions

Many **linear differential equations** that occur in mathematical physics (as, e.g., **Laplace's equation** for steady irrotational flow to be discussed below) give rise to solutions that can be represented as expansions over **complete sets of orthonormal functions** on a given interval. The particular complete set that one uses depends on the geometry of the problem at hand.

Consider, in *one* spatial dimension, an interval (a,b) in a variable x, with a complete set of real or complex functions $U_n(x)$, $n = 1,2,\ldots,\infty$, that are square integrable and orthonormal on (a,b). That is, they satisfy the

Orthonormality condition:

$$\int_a^b U_n^*(x)\,U_m(x)\,\mathrm{d}x = \delta_{nm}\;, \tag{2.60}$$

with δ_{nm} denoting the **Kronecker symbol**, and the

Completeness condition:

$$\sum_{n=1}^{\infty} U_n^*(x')\,U_n(x) = \delta(x'-x) \ , \tag{2.61}$$

with $\delta(x'-x)$ denoting **Dirac's delta function**. Then an arbitrary real-valued function $f(x)$ that is square integrable on (a,b) can be expanded in an infinite series of the $U_n(x)$ according to

$$f(x) = \sum_{n=1}^{\infty} a_n\,U_n(x) \tag{2.62}$$

$$a_n = \int_a^b U_n^*(x)\,f(x)\,\mathrm{d}x \ . \tag{2.63}$$

The constant expansion coefficients a_n are said to represent the **spectrum** of $f(x)$ with respect to the $U_n(x)$.

That the right-hand side of (2.62) does indeed provide a rigorous representation of $f(x)$ can be seen as follows. Using (2.63) in (2.62), we find

$$f(x) = \sum_{n=1}^{\infty} \left[\int_a^b U_n^*(x')\,f(x')\,\mathrm{d}x' \right] U_n(x) \ ,$$

the right-hand side of which can be rewritten as

$$\sum_{n=1}^{\infty} \left[\int_a^b U_n^*(x')\,f(x')\,\mathrm{d}x' \right] U_n(x) = \int_a^b f(x') \left[\sum_{n=1}^{\infty} U_n^*(x')\,U_n(x) \right] \mathrm{d}x' \ .$$

But then, by (2.61), we have

$$\int_a^b f(x') \left[\sum_{n=1}^{\infty} U_n^*(x')\,U_n(x) \right] \mathrm{d}x' = \int_a^b f(x')\,\delta(x'-x)\,\mathrm{d}x' = f(x) \ ,$$

which ends this demonstration.

Note that the concept of orthonormal function expansions can be conveniently extended to square integrable functions in *three* spatial dimensions.

2.9.1 Fourier series expansions

The *most famous* set of orthonormal functions are the normalised sines and cosines,

$$\sqrt{\frac{2}{a}}\,\sin\left(\frac{n\,2\pi}{a}\,x\right) \ , \quad \sqrt{\frac{2}{a}}\,\cos\left(\frac{n\,2\pi}{a}\,x\right) \ , \qquad n = 1, 2, \ldots, \infty$$

used in a **Fourier series expansion** (named after the French mathematician Jean Baptiste Joseph Fourier, 1768–1830) of a real-valued function $f(x)$ that is periodic over an x–interval $(-a/2, a/2)$. Namely,

$$\begin{aligned} f(x) &= \tfrac{1}{2}\,A_0 + \sqrt{\frac{2}{a}}\sum_{n=1}^{\infty}\left[A_n\,\cos\left(\frac{n\,2\pi}{a}\,x\right) + B_n\,\sin\left(\frac{n\,2\pi}{a}\,x\right)\right] \\ A_n &= \sqrt{\frac{2}{a}}\int_{-a/2}^{a/2} f(x)\,\cos\left(\frac{n\,2\pi}{a}\,x\right)\mathrm{d}x \\ B_n &= \sqrt{\frac{2}{a}}\int_{-a/2}^{a/2} f(x)\,\sin\left(\frac{n\,2\pi}{a}\,x\right)\mathrm{d}x\;. \end{aligned}$$

Alternatively, using the normalised complex exponentials

$$U_n(x) = \frac{1}{\sqrt{a}}\,\mathrm{e}^{i\frac{n\,2\pi}{a}\,x}\;, \quad n = 0, \pm 1, \pm 2, \dots, \pm\infty\;,$$

$f(x)$ can be expressed by the expansion

$$f(x) = \frac{1}{\sqrt{a}}\sum_{n=-\infty}^{+\infty} A_n\,\mathrm{e}^{i\frac{n\,2\pi}{a}\,x} \tag{2.64}$$

$$A_n = \frac{1}{\sqrt{a}}\int_{-a/2}^{a/2}\mathrm{e}^{-\,i\frac{n\,2\pi}{a}\,x}\,f(x)\,\mathrm{d}x\;. \tag{2.65}$$

2.9.2 Fourier integral representations

In the *limit* that we let the x–interval $(-a/2, a/2)$ become infinite, i.e., $a \to \infty$, while simultaneously making the transitions

$$\begin{aligned} \frac{n\,2\pi}{a}\,x &\longrightarrow k \\ \sum_{n=-\infty}^{+\infty} &\longrightarrow \int_{-\infty}^{\infty}\mathrm{d}n = \frac{a}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}k \\ A_n &\longrightarrow \sqrt{\frac{2\pi}{a}}\,A(k) \end{aligned}$$

from discretely to continuously varying quantities, we obtain the **Fourier integral representation** of a square integrable real-valued function $f(x)$ (“of period infinity”) over the interval $(-\infty, +\infty)$ given by

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)\,\mathrm{e}^{ikx}\,\mathrm{d}k \tag{2.66}$$

$$A(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-ikx}\,f(x)\,\mathrm{d}x\;. \tag{2.67}$$

The orthonormality condition for the continuous set of square integrable functions $U(k,x) = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{ikx}$ on $(-\infty, +\infty)$ reads

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{i(k-k')x}\,\mathrm{d}x = \delta(k-k')\;, \tag{2.68}$$

while the completeness condition reads

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{ik(x-x')x}\,\mathrm{d}k=\delta(x-x')\ . \tag{2.69}$$

Hence, both of these integral expressions provide convenient representations of **Dirac's delta function** discussed in section 2.8.

We conclude by stating that straightforward extensions of the **Fourier integral representation** to square integrable functions in *three* spatial dimensions do exist.

Chapter 3
Introduction to fluid flows

3.1 Basic concepts

The aim of this section is to introduce some of the most important equations and concepts of **fluid dynamics**, so that we can begin learning about fluid flows as soon as possible. We will return to many of these topics in more detail, and more rigorously, later.

Provided that there are *no* chemical or nuclear reactions happening, then a very important physical principle is that of **conservation of mass**. In other words, for a given body of fluid the amount of matter in it remains constant.

Another important concept when it comes to describing fluids is that of **compressibility**, which is, loosely speaking, how "squeezable" the fluid is. For example, water (the fluid which we are most familiar with) is nearly incompressible, and this imposes important restrictions on the types of flow that are possible. Air, on the other hand, is more easily compressed.

But first we discuss the relationship between the **Eulerian description** of the variation of fluid quantities at fixed spatial positions and the **Lagrangian description** of the variation of fluid quantities as one moves along with a particular fixed fluid element.

3.1.1 Convective derivative

In general in this course we want to use the **Eulerian description** of **fluid dynamics** (as, e.g., we have done when describing streamlines). But some fluid quantities are best expressed in **Lagrangian terms**, e.g., conserved quantities such as mass or linear momentum. Thus we want to relate the two formulations. In particular, we want to relate the different concepts of "rate of change" in the two descriptions.

Choosing **Cartesian coordinates**, consider a function $f(t, x, y, z)$ as representing some quantity describing a particular aspect of the motion of a fluid (such as mass density or a component of the flow velocity). The partial derivative $\partial f/\partial t$ means the rate of change of f with respect to t at *fixed spatial position* $\boldsymbol{r} = (x, y, z)^T$.

We can also find the rate of change of the quantity f as we "follow the fluid", namely, how f changes as we track a particular **Lagrangian fluid element**. This is called the **convective derivative** (or, in some textbooks, the material derivative, Lagrangian derivative, or mobile derivative). We denote this by

$$\left.\frac{\mathrm{D}f}{\mathrm{D}t}\right|_{\boldsymbol{q}} = \frac{\mathrm{d}}{\mathrm{d}t} f[t, x(t), y(t), z(t)] \;, \tag{3.1}$$

where now the position coordinates $\{x(t), y(t), z(t)\}$ of a *fixed* **Lagrangian fluid element** labelled $\boldsymbol{q}$ are considered as functions of time which describe the motion of the fluid element. That is, these position coordinates change with time as the local flow velocity $\boldsymbol{u}$. The function f is in Eulerian terms, but the position coordinates $\{x, y, z\}$ are of the Lagrangian kind; they are *no longer* the independent variables. Thus,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u_x \;, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = u_y \;, \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = u_z \;. \tag{3.2}$$

So, for example, the rate of change of the x-position of a fluid element with label $\boldsymbol{q}$ is the x-component of the flow velocity $\boldsymbol{u}$.

The chain rule of differentiation then gives for the convective derivative of f

$$\left.\frac{\mathrm{D}f}{\mathrm{D}t}\right|_{\boldsymbol{q}} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} \;, \tag{3.3}$$

i.e.,

$$\left.\frac{\mathrm{D}f}{\mathrm{D}t}\right|_{\boldsymbol{q}} = \frac{\partial f}{\partial t} + u_x\frac{\partial f}{\partial x} + u_y\frac{\partial f}{\partial y} + u_z\frac{\partial f}{\partial z} \;. \tag{3.4}$$

In vector form this can be written as

$$\left.\frac{\mathrm{D}f}{\mathrm{D}t}\right|_{\boldsymbol{q}} = \left.\frac{\partial f}{\partial t}\right|_{\boldsymbol{r}} + \left.(\boldsymbol{u}\cdot\nabla)f\right|_t \;. \tag{3.5}$$

What this says is that the Lagrangian rate of change in time of any fluid quantity f for a particular fixed fluid element (labelled $\boldsymbol{q}$) is made up of *two parts*: the rate of change in time of f *at the instantaneous spatial position* of the fluid element *and* the rate of change of f due to the fact that the fluid element is moving from one place to another, where f may have a *different value* at a given instant in time. Thus the convective derivative gives the Lagrangian rate of change in time of f in terms of Eulerian measurements. Note that in this derivation we have *neglected* second-order terms due to, e.g., the time variation of f at the new position of the particle. Unfortunately, the second contribution to $\mathrm{D}f/\mathrm{D}t$ is sometimes (confusingly!) called the convective derivative. A better way is to speak of the "convective term".

An alternative derivation is the following. Consider any fluid quantity $f(t, \boldsymbol{r})$. The difference in f at two different times t and $t + \Delta t$, for a given **Lagrangian fluid element** labelled $\boldsymbol{q}$, is

$$f(t + \Delta t, \boldsymbol{q}) - f(t, \boldsymbol{q}) = \left.\frac{\mathrm{D}f}{\mathrm{D}t}\right|_{\boldsymbol{q}} \Delta t + O(\Delta t^2) \;. \tag{3.6}$$

But, if the position of the fluid element is $\boldsymbol{r}$ at time t and $\boldsymbol{r} + \boldsymbol{\Delta r}$ at $t + \Delta t$, then this difference can be written relative to an **Eulerian reference frame** as

$$f(t + \Delta t, \boldsymbol{r} + \boldsymbol{\Delta r}) - f(t, \boldsymbol{r}) = f(t + \Delta t, \boldsymbol{r} + \boldsymbol{\Delta r}) - f(t + \Delta t, \boldsymbol{r})$$

$$+ f(t+\Delta t, \boldsymbol{r}) - f(t, \boldsymbol{r})$$

$$= \boldsymbol{\Delta r} \cdot \nabla f|_{t+\Delta t}\, \Delta t + \left.\frac{\partial f}{\partial t}\right|_{\boldsymbol{r}} \Delta t + O(\Delta t^2, \boldsymbol{\Delta r}^2)\ , \tag{3.7}$$

using Taylor expansions of f in both t and $\boldsymbol{r}$. So, neglecting higher-order terms, equating (3.6) and (3.7) yields

$$\left.\frac{\mathrm{D}f}{\mathrm{D}t}\right|_{\boldsymbol{q}} = \left.\frac{\partial f}{\partial t}\right|_{\boldsymbol{r}} + \frac{\boldsymbol{\Delta r}}{\Delta t} \cdot \nabla f|_{t+\Delta t}\ . \tag{3.8}$$

In the limit $\Delta t \to 0$ this becomes

$$\left.\frac{\mathrm{D}f}{\mathrm{D}t}\right|_{\boldsymbol{q}} = \left.\frac{\partial f}{\partial t}\right|_{\boldsymbol{r}} + (\boldsymbol{u} \cdot \nabla) f|_t\ , \tag{3.9}$$

since the rate of change of a fluid element's position at a certain position and time is the flow velocity at that position and time. To maintain notational transparency we will from now on drop the label $\boldsymbol{q}$ from the convective derivative.

As examples let us consider how we can calculate **flow velocity** (which is the rate of change in time of spatial position as one moves with a fixed fluid element) — this is

$$\frac{\mathrm{D}\boldsymbol{r}}{\mathrm{D}t} = (\boldsymbol{u} \cdot \nabla)\boldsymbol{r} = \boldsymbol{u}\ , \tag{3.10}$$

and **flow acceleration** (which is the rate of change in time of flow velocity as one moves with a fixed fluid element) — this is

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \boldsymbol{a} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}\ , \tag{3.11}$$

[cf. (1.43)]. It is clear, then, that $\mathrm{D}\boldsymbol{u}/\mathrm{D}t$ can be non-zero even in steady flows. In a **steady flow**, if the *direction* of the flow velocity changes, then a fluid element has to accelerate in order to follow the twists and turns of the flow pattern.

Consider a 2-D flow describing a fluid in uniform rotation at angular velocity Ω, so that in **Cartesian coordinates**

$$\boldsymbol{u} = (-\,\Omega y, \Omega x, 0)^T\ . \tag{3.12}$$

Since $\partial \boldsymbol{u}/\partial t = \boldsymbol{0}$ (as the flow is steady), the flow acceleration is

$$\begin{aligned}(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} &= \left(-\,\Omega y\, \frac{\partial}{\partial x} + \Omega x\, \frac{\partial}{\partial y}\right)(-\,\Omega y, \Omega x, 0)^T \\ &= -\,\Omega^2 (x, y, 0) = -\,\Omega^2\, \boldsymbol{r}\ .\end{aligned} \tag{3.13}$$

This is just the centrifugal acceleration $\Omega^2 r$ directed towards the rotation axis.

Constant along a streamline

In a **steady flow**, the rate of change of some scalar-valued fluid quantity $f(\boldsymbol{r})$ following a particular fluid element is just $(\boldsymbol{u} \cdot \nabla) f$. Thus the relation

$$(\boldsymbol{u} \cdot \nabla) f(\boldsymbol{r}) = 0 \tag{3.14}$$

implies that in a steady flow f is *constant along a streamline*. Of course, f may take different values on different streamlines; f is not necessarily constant throughout the flow.

Constant for a fluid element

Similarly, the relation

$$\frac{\mathrm{D}f(t,\boldsymbol{r})}{\mathrm{D}t} = \frac{\partial f(t,\boldsymbol{r})}{\partial t} + (\boldsymbol{u}\cdot\nabla)f(t,\boldsymbol{r}) = 0 \tag{3.15}$$

for some scalar-valued fluid quantity $f(t,\boldsymbol{r})$ implies that f is *constant for a particular fluid element*, i.e., constant along its particle path. Of course, f may have different values for different fluid elements.

3.1.2 Balance equations

One of the unifying concepts in theoretical physics is the observation that dynamical interactions between two coupled physical systems are best described in terms of the exchange of physical quantities that flow from one system to the other, or vice versa. Let us discuss this issue on a more abstract level.

Let X represent a **physical quantity** whose magnitude scales with the **volume** (i.e., size) of a physical system. This is to say that the value of X doubles when the volume of the system doubles. A physical quantity with this property is referred to as **extensive**.[1] It distinguishes itself through the feature that it can form **densities** with respect to volumes and **current densities** with respect to surfaces. Note that extensive quantities can be **scalar-valued** or **vector-valued**. Examples of scalar-valued extensive quantities are **mass**, **electric charge**, **particle number**, **energy** or **entropy**, while examples of vector-valued physical quantities are **linear momentum** or **angular momentum**.

Let us consider in Euclidian space $\mathbb{R}^3$ a **volume** G, which is bounded by a **closed surface** ∂G. By assumption, this bounding surface shall be fixed in time. Then, if X is an extensive physical quantity, the

$$\begin{pmatrix}\textbf{Rate of change}\\ \textbf{in time}\text{ of the}\\ \text{amount of } X\\ \text{inside of } G\end{pmatrix} = \begin{pmatrix}\textbf{Current } I_X\\ \text{of } X \text{ into } G\end{pmatrix} + \begin{pmatrix}\textbf{Generation rate } \Sigma_X\\ \text{of } X \text{ inside of } G\end{pmatrix},$$

stating that for X the

Balance equation:

$$\frac{\mathrm{d}X}{\mathrm{d}t} = I_X + \Sigma_X \tag{3.16}$$

[1]Physical quantities, on the other hand, that do *not* have this property are referred to as intensive. Examples of intensive quantities are mass density, pressure or temperature.

holds. Now if for a *scalar-valued* X we introduce as differentiable functions of time and spatial position the scalar-valued X–**density** $\rho_X = \rho_X(t, \boldsymbol{r})$, the vector-valued X–**current density** $\boldsymbol{j}_X = \boldsymbol{j}_X(t, \boldsymbol{r})$, and the scalar-valued X–**generation rate density** $\dot{\rho}_{X,\text{gen}} = \dot{\rho}_{X,\text{gen}}(t, \boldsymbol{r})$, then the balance equation (3.16) for X can be rewritten in the **integral form**

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_G \rho_X \,\mathrm{d}V = -\iint_{\partial G} \boldsymbol{j}_X \cdot \boldsymbol{n}\,\mathrm{d}A + \iiint_G \dot{\rho}_{X,\text{gen}}\,\mathrm{d}V\ , \quad (3.17)$$

with $\boldsymbol{n}$ denoting the outward-pointing **unit normal** to the closed surface ∂G. The *minus sign* of the flux integral on the right-hand side arises because we record the current I_X that flows *into* the volume G. If we now employ Gauß' integral theorem (2.32) to this flux integral, and then bring all terms in the equation to one side, we get the balance equation for X in the form

$$0 = \iiint_G \left[\frac{\partial \rho_X}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j}_X - \dot{\rho}_{X,\text{gen}}\right]\mathrm{d}V\ .$$

Note that, as we assumed G to be fixed in time, we are allowed to pull the total time derivative through the volume integral sign; consequently under the integral sign this time derivative becomes a partial time derivative.

Now since our balance equation must hold for any arbitrary volume G of $\mathbb{R}^3$ that is fixed in time, it is the integrand in the expression we just derived that must vanish identically. We thus find the **differential form** of the balance equation for X to be given by

$$\frac{\partial \rho_X}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j}_X = \dot{\rho}_{X,\text{gen}}\ . \quad (3.18)$$

It should be pointed out that when X is a *vector-valued* extensive physical quantity, then an analogous balance equation can be formulated in terms of a vector-valued X–**density**, a tensor-valued X–**current density**, and a vector-valued X–**generation rate density**.

Remark: If for a specific physical quantity X empirical results show that the X–generation rate density $\dot{\rho}_{X,\text{gen}}$ is *identically zero* (such as for mass, electric charge, energy, linear momentum and angular momentum), then (3.17) and (3.18) are referred to as **continuity equations** or **conservation equations**.

Physical dimensions:

$$[\,\rho_X\,] = \frac{[X]}{[\text{length}]^3} \qquad [\,\boldsymbol{j}_X\,] = \frac{[X]}{[\text{length}]^2[\text{time}]} \qquad [\,\dot{\rho}_{X,\text{gen}}\,] = \frac{[X]}{[\text{length}]^3[\text{time}]}\ .$$

3.1.3 Conservation of mass

In this course we will *not* deal with exotic fluids which spontaneously generate or destroy material, and, for the moment, we are not concerned with any regions with sources or sinks of materials (e.g., taps and plug-holes). In these circumstances, using the concepts just introduced in subsection 3.1.2, we will derive a very important relation which will often be applicable, regardless of the type of fluid under consideration.

The **mass** of fluid within a closed surface ∂G bounding a *fixed* volume G is given by $m = \int\int\int_G \rho \, \mathrm{d}V$, where $\rho(t, \boldsymbol{r})$ is the mass density. The rate of change in time of this mass is then

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \int\int\int_G \frac{\partial \rho}{\partial t} \, \mathrm{d}V \, . \tag{3.19}$$

If *no* matter is created/added or destroyed/removed within G, then this rate of change can only be due to fluid moving across ∂G. The net **mass inflow rate** is $-\int\int_{\partial G} \rho \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{A}$, since $\mathrm{d}\boldsymbol{A}$ points *outward* from the volume. As this must balance $\mathrm{d}m/\mathrm{d}t$, it follows that

$$\int\int\int_G \frac{\partial \rho}{\partial t} \, \mathrm{d}V + \int\int_{\partial G} \rho \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{A} = 0 \, . \tag{3.20}$$

Using Gauß' integral theorem (2.32) thus gives

$$\int\int\int_G \left[\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) \right] \mathrm{d}V = 0 \, , \tag{3.21}$$

and, since the volume we choose to integrate over is arbitrary and we assume that the fluid variables are *continuous*, it follows that

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0 \, . \tag{3.22}$$

The Leibniz rule of differentiation shows $\boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = \boldsymbol{u} \cdot \boldsymbol{\nabla} \rho + \rho \, (\boldsymbol{\nabla} \cdot \boldsymbol{u})$, and when we employ the definition of the convective derivative given in (3.5), we get in the **Lagrangian description**

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \, (\boldsymbol{\nabla} \cdot \boldsymbol{u}) = 0 \, . \tag{3.23}$$

This [or in the form (3.22)] is the **equation of conservation of mass**, also known as the **continuity equation**.

3.1.4 Incompressibility

Consider a fixed, closed surface ∂G in a fluid, with outward normal $\boldsymbol{n}$. As the fluid flows, some parts of ∂G will have fluid entering, and other parts will have fluid exiting. The volume of fluid leaving through a small surface element $\boldsymbol{n} \, \mathrm{d}A$ in unit time will be $\boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}A$, giving us the **volume flux** of fluid through $\mathrm{d}A$ along its *normal direction* (remember velocity times area has physical dimension $[\,\text{volume}\,]\,[\,\text{time}\,]^{-1}$, i.e., volume flux, and we are only interested in the normal volume flux). So the net volume rate of fluid that is leaving ∂G is

$$\int\int_{\partial G} \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}A \, . \tag{3.24}$$

For an **incompressible fluid** this must be *zero* (one cannot squeeze any more material into ∂G), so, using Gauß' integral theorem (2.32), we conclude that

$$\int\int\int_G \boldsymbol{\nabla} \cdot \boldsymbol{u} \, \mathrm{d}V = 0 \, . \tag{3.25}$$

This must be true for all possible volumes G in the fluid. Hence, we can conclude that

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \, . \tag{3.26}$$

This is the **incompressibility condition**, and it is widely applicable to fluids (viscous or inviscid). Most of this course studies fluids for which this condition is true. Note that from (3.23) it follows that for incompressible fluids

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0 \, , \tag{3.27}$$

(employing the **Lagrangian description**).

3.1.5 Pressure force

In a **static** fluid (i.e., when $\boldsymbol{u} = \boldsymbol{0}$), for every fluid element the **force per unit area** that it feels due to the molecular mechanical interactions with its neighbours has the *same* magnitude in *all* directions, or, in other words, it is **isotropic**. We identify this force per unit area with the fluid's **mechanical pressure**, also known as **normal stress**. This observation is the basis of **Pascal's principle**:

> *Pressure applied to an enclosed fluid is transmitted undiminished to every portion of the fluid and to the walls of the containing vessel.*

Consider a surface element $\boldsymbol{n} \, \mathrm{d}A$ where $\boldsymbol{n}$ is the **unit normal** to the surface element. Since the pressure is isotropic, the pressure force is in the normal direction:[2]

$$p \, \boldsymbol{n} \, \mathrm{d}A \, . \tag{3.28}$$

This is the force on the fluid in the direction of $\boldsymbol{n}$.

Now consider a closed surface ∂G enclosing a **blob of fluid**. The pressure exerted by the *surrounding* fluid across a surface element $\mathrm{d}A$ of ∂G is $-p \, \boldsymbol{n} \, \mathrm{d}A$ because, by convention, we choose the normal to point outwards from the surface ∂G. Remember that *at* the surface element there are equal and opposite

[2]Typically we will want to denote the mechanical pressure by P, and thus distinguish it from the thermodynamical pressure, p. However, for static fluids (when $\boldsymbol{u} = \boldsymbol{0}$), incompressible fluids (when $\boldsymbol{\nabla} \cdot \boldsymbol{u} = \boldsymbol{0}$), or inviscid fluids the two are identical.

forces on the fluid on either side of the surface element. So the net force on the fluid blob is found by integrating this force over its entire surface. Thus,

$$-\int\!\!\int_{\partial G} p\,\boldsymbol{n}\,\mathrm{d}A = -\int\!\!\int\!\!\int_G \boldsymbol{\nabla} p\,\mathrm{d}V\ . \tag{3.29}$$

If $\boldsymbol{\nabla} p$ is continuous, then it will be approximately constant over the volume of the blob, provided that the blob is sufficiently small. We can then deduce that the net force felt by a small fluid volume $\mathrm{d}V$ (as exerted on the blob by the rest of the fluid) will be $-\,\boldsymbol{\nabla} p\,\mathrm{d}V$. Alternatively, the pressure force per unit volume is

$$\boldsymbol{F}_p = -\,\boldsymbol{\nabla} p\ . \tag{3.30}$$

Above we used the *identity*

$$\int\!\!\int_{\partial G} \phi\,\boldsymbol{n}\,\mathrm{d}A = \int\!\!\int\!\!\int_G \boldsymbol{\nabla}\phi\,\mathrm{d}V\ , \tag{3.31}$$

which can be proved from Gauß' integral theorem (2.32), starting from the vector $\phi\,\boldsymbol{a}$, where $\boldsymbol{a}$ is some *constant* vector.

3.1.6 Hydrostatics and Archimedes' principle

For fluids in a **gravitational field** (as on the Earth where it is effectively *constant*) each fluid element is also acted on by a gravitational force. This is an example of what is called a **body force** since it acts throughout the body of the fluid, not just on its edges (such as surfaces). If the (constant) **gravitational acceleration** is $\boldsymbol{g}$, then the gravitational force per unit volume, for a fluid of mass density ρ, is

$$\boldsymbol{F}_g = \rho\,\boldsymbol{g}\ . \tag{3.32}$$

This is like the gravitational force $m\,\boldsymbol{g}$ on a point particle of mass m.

For a fluid that is **static** (i.e., $\boldsymbol{u} = \boldsymbol{0}$), consider a small volume of fluid, $\mathrm{d}V$. The sum of all forces acting on it must be zero, i.e., the pressure force per unit volume and the gravitational force per unit volume must balance. This requirement thus leads us [by equating (3.30) and (3.32)] to the

Equation of hydrostatic equilibrium:

$$\boldsymbol{F}_p + \boldsymbol{F}_g = -\boldsymbol{\nabla} p + \rho\,\boldsymbol{g} = \boldsymbol{0}\ . \tag{3.33}$$

It shows that the pressure *increases* in the direction of $\boldsymbol{g}$. It also follows that the denser the fluid, then the higher the pressure gradient, and that any horizontal plane in the fluid is a surface of constant pressure. We can integrate the equation of hydrostatic equilibrium by using the scalar-valued Newtonian **gravitational potential** Φ defined by $\boldsymbol{g} := -\boldsymbol{\nabla}\Phi$, and if in **Cartesian coordinates** $\boldsymbol{g} = -\,g\,\boldsymbol{e}_z$, $g =$ constant > 0 (with the z–axis pointing upward), then $\Phi = g\,z$. So, assuming *constant* mass density (such that the fluid is incompressible),

$$-\,\boldsymbol{\nabla} p - \rho\boldsymbol{\nabla}\Phi = -\,\boldsymbol{\nabla}(p + \rho\,\Phi) = \boldsymbol{0}\ , \tag{3.34}$$

which, in component form, can be written as

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0 \; , \qquad \frac{\partial}{\partial z}(p + \rho\, g\, z) = 0 \; . \tag{3.35}$$

It follows that p is independent of x and y, and that

$$p + \rho\, g\, z = \text{constant} \; . \tag{3.36}$$

Alternatively, suppose there is a single body of fluid. Then at two different heights within it, z_1 (where the pressure is p_1) and z_2 (where the pressure is p_2), we have the following relationship:

$$p_1 + \rho\, g\, z_1 = p_2 + \rho\, g\, z_2 \; . \tag{3.37}$$

This is sometimes expressed as

$$p_2 - p_1 = \rho\, g\, h \; , \tag{3.38}$$

where $h := z_1 - z_2$ is the difference in height within the fluid between the two positions z_1 and z_2.

Now we move on to the situation of a **body** which is totally immersed in a static fluid in a uniform gravitational field. For simplicity we assume again that the fluid has constant mass density. The body shall displace fluid in a domain G, of total volume V, which is enclosed by a solid surface ∂G. The total pressure force acting on the body, the **buoyant force** $\boldsymbol{F}_b$, can be found by integrating the pressure over the surface of the body, and we can use the same procedure as above to turn this into a volume integral over the volume of the fluid displaced by the body:

$$\begin{aligned} \boldsymbol{F}_b &= -\int\!\!\int_{\partial G} p\, \boldsymbol{n}\, \mathrm{d}A & (3.39) \\ &= -\int\!\!\int\!\!\int_G \boldsymbol{\nabla} p\, \mathrm{d}V & (3.40) \\ &= -\int\!\!\int\!\!\int_G \rho\, \boldsymbol{g}\, \mathrm{d}V & (3.41) \\ &= \rho\, V\, g\, \boldsymbol{e}_z \; , & (3.42) \end{aligned}$$

with V the total volume of G. Here we have used the equation of hydrostatic equilibrium and the constancy of ρ and g over G. This result states that:

The pressure force (buoyant force) on an immersed body is equal and opposite to the total force of gravity acting on the displaced fluid.

This is the well known **Archimedes' principle**, named in honour of the ancient Greek philosopher Archimedes (287BC–212BC).

3.2 Ideal fluids

When making mathematical models for describing natural phenomena, it is always helpful to simplify, and in **fluid dynamics** the greatest simplification is to assume that the *viscosity is zero*: this is the basis for what is called an "ideal fluid". One of the ways to characterize such a fluid is as "dry water" (!), which shows that ideal does not mean realistic!

An **ideal fluid** has the following properties:

- The only force exerted across a geometrical surface element $\boldsymbol{n}\,\mathrm{d}A$ within the fluid is the (mechanical) **pressure force**

$$p\,\boldsymbol{n}\,\mathrm{d}A\,, \tag{3.43}$$

  where the pressure p is a scalar-valued function independent of the **unit normal** $\boldsymbol{n}$. It follows that an ideal fluid is **inviscid**. (The force is exerted on the fluid into which $\boldsymbol{n}$ is pointing.)

- The fluid is **isothermal**, i.e., the scalar-valued temperature T is independent of time t and the same for all fluid elements, so $T =$ constant.

We emphasise that in our applications in this course of **ideal fluids** we will regularly make the additional assumptions that:

- The fluid is **incompressible**, so that no finite sized fluid blob changes volume as it moves with the flow.

- The fluid has **constant mass density**, i.e., it is independent of time t and the *same* for all fluid elements, so $\rho =$ constant.

3.2.1 Euler's equations of motion

We wish to derive **equations of motion** applicable to an **ideal fluid** which is incompressible and has constant mass density. We are quite familiar with the equation of motion for a point particle of constant mass (e.g., tennis ball) that is derived using **Newton's equations of motion** (or **conservation of linear momentum**): the rate of change in time of the particle's linear momentum equals the force (momentum current) acting on it; frequently this is turned into "force equals mass times acceleration", as stated in (1.2). But in the case of a fluid we must consider the motion of the same Lagrangian **blob of fluid**. In other words, we have to consider a blob given by a small volume of fluid, $\mathrm{d}V$, and it is *this* which is analogous to a particle. The flow acceleration felt by this blob is then derived using the convective derivative to find the rate of change in time of the blob's flow velocity *as it moves with the fluid*. We have just found an expression for the force on a blob of fluid due to pressure variations in a fluid. If the only other body force is gravitational ($\boldsymbol{g}$, force per unit mass), the equation of motion for a blob of fluid, of mass density $\rho =$ constant and, hence, of constant mass $\rho\,\mathrm{d}V$, is

$$\frac{\mathrm{D}}{\mathrm{D}t}(\rho\,\boldsymbol{u}\,\mathrm{d}V) = \rho\,\mathrm{d}V\,\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = (-\boldsymbol{\nabla}p + \rho\,\boldsymbol{g})\,\mathrm{d}V\ , \tag{3.44}$$

(employing the **Lagrangian description**). Thus, the motion of an incompressible ideal fluid is described by the two equations known as

Euler's equations:

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\frac{1}{\rho}\,\boldsymbol{\nabla}p + \boldsymbol{g} \tag{3.45}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0\ , \tag{3.46}$$

which the Swiss mathematician Leonhard Euler (1707–1783) derived in 1755.

If one orients the z–direction of a **Cartesian reference frame** "upwards", then near the surface of the Earth the **gravitational acceleration** is $\boldsymbol{g} = (0, 0, -g)$, $g =$ constant > 0. But as (Newtonian) **gravity** is a **conservative force**, the gravitational acceleration may be expressed as the gradient of a scalar-valued **gravitational potential** Φ,

$$\boldsymbol{g} = -\boldsymbol{\nabla}\Phi\ . \tag{3.47}$$

Thus, with $\rho =$ constant and (3.5), (3.45) can be written as

$$\frac{\partial\boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} = -\,\boldsymbol{\nabla}\left(\frac{p}{\rho} + \Phi\right)\ . \tag{3.48}$$

A further algebraic manipulation is possible: using the vector analytical identity

$$(\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V} = \tfrac{1}{2}\boldsymbol{\nabla}(\boldsymbol{V}\cdot\boldsymbol{V}) + (\boldsymbol{\nabla}\times\boldsymbol{V})\times\boldsymbol{V}\ , \tag{3.49}$$

[cf. (2.30)], together with the definition $\boldsymbol{\omega} := \boldsymbol{\nabla}\times\boldsymbol{u}$ of the **fluid vorticity**, Euler's momentum equation (3.45) can finally be written in the form

$$\frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{\omega}\times\boldsymbol{u} = -\,\boldsymbol{\nabla}\left(\tfrac{1}{2}\,u^2 + \frac{p}{\rho} + \Phi\right)\ ; \tag{3.50}$$

here we also employed the definition $u^2 := \boldsymbol{u}\cdot\boldsymbol{u}$ for the squared flow velocity magnitude.

3.2.2 Bernoulli's streamline theorem

When we deal with a **steady flow** (so that $\partial\boldsymbol{u}/\partial t = \boldsymbol{0}$), Euler's momentum equation (3.50) reduces to

$$\boldsymbol{\omega}\times\boldsymbol{u} = -\,\boldsymbol{\nabla}H\ ; \tag{3.51}$$

the quantity H is called the **specific gravito-enthalpy**[3] and is defined as

$$H := \tfrac{1}{2}\,u^2 + \frac{p}{\rho} + \Phi\ . \tag{3.52}$$

[3]More precisely, the specific gravito-enthalpy is $H = \frac{1}{2}\,u^2 + \epsilon + p/\rho + \Phi$, where ϵ is the specific internal energy of the fluid due to internal molecular motions and forces. However, as we implicitly assume $\epsilon =$ constant, we can safely neglect it here.

Its physical dimension is $[\,\text{velocity}\,]^2 = [\,\text{length}\,]^2\,[\,\text{time}\,]^{-2}$, and so it has SI unit 1 $\text{m}^2\ \text{s}^{-2}$.

Now taking the dot product of (3.51) with $\boldsymbol{u}$, the left-hand side becomes identically zero and we obtain

$$(\boldsymbol{u}\cdot\boldsymbol{\nabla})H = 0\,. \tag{3.53}$$

We have thus proved for steady flow **Bernoulli's streamline theorem**, which can be expressed by

Bernoulli's equation:

$$H = \tfrac{1}{2}\,u^2 + \frac{p}{\rho} + \Phi = \text{constant along streamline}\,. \tag{3.54}$$

This is a relation of fundamental practical importance. It is named after the Swiss mathematician, physician, and physicist Daniel Bernoulli (1700–1782).

In words **Bernoulli's streamline theorem** states:

If an incompressible ideal fluid is in steady flow, then the specific gravito-enthalpy H is constant along every streamline.

Note that *different* streamlines can have *different* values of H associated with them.

3.2.3 Bernoulli's streamline theorem for irrotational flow

If the flow is **steady** *and* **irrotational**, so that in addition to $\partial\boldsymbol{u}/\partial t = \boldsymbol{0}$ also $\boldsymbol{\omega} = \boldsymbol{0}$ holds, then (3.51) reduces to $\boldsymbol{\nabla}H = \boldsymbol{0}$. In this case, the specific gravito-enthalpy H is independent of position and time, i.e.,

$$H = \tfrac{1}{2}\,u^2 + \frac{p}{\rho} + \Phi = \text{constant}\,. \tag{3.55}$$

In words:

If an incompressible ideal fluid is in steady, irrotational flow, then the specific gravito-enthalpy H is constant throughout the fluid.

3.2.4 Example: fluid streams

One of the simplest inviscid flows is that of a stream of fluid (such as water) flowing through a stationary fluid with which it does not mix (such as air). Consider water running from a tap:

(If you do this experiment, keep the rate of flow quite slow, otherwise various ripple instabilities disturb the surface of the flow; you need to have a **smooth** (laminar) and **cylindrically symmetric flow**.) The water leaves the tap at position z_1 with a speed u_1 as a circular stream of diameter d_1. As the water falls subject to a constant gravitational acceleration, $\boldsymbol{g}$, the stream speeds up and gradually narrows. Eventually the stream becomes so thin that **surface tension** forces (another fluid force; one that we have *not* discussed) become important and the stream breaks up into a series of droplets. The stage before breakup can be described using **Bernoulli's streamline theorem** as applied to the central streamline of the water stream.

Choosing a lower position z_2 in the stream and labelling quantities there by "2", we have

$$\frac{u_2^2}{2} + \frac{p_2}{\rho} + g\,z_2 = \frac{u_1^2}{2} + \frac{p_1}{\rho} + g\,z_1\ . \tag{3.56}$$

The position of the surface of the stream, which does not move, is based on pressure balance between the pressure in the water and the pressure of the surrounding air. Thus the values of the pressure in the water at the centre of the stream is the *same* as the pressure of the **atmosphere** at the *same* height z, i.e., $p(z) = p_{\rm atm}(z)$. We are *assuming* that radial acceleration of the water stream is negligible, and that surface tension is not important. Thus the values of pressure in the water at positions 1 and 2 are related by the **equation of hydrostatic equilibrium** for the atmosphere (with $p(z) = p_{\rm atm}(z)$), giving

$$p_2 + \rho_{\rm atm}\,g\,z_2 = p_1 + \rho_{\rm atm}\,g\,z_1\ , \tag{3.57}$$

where $\rho_{\rm atm}$ is the mass density of the surrounding air which we can assume to be constant. Substituting, we find for the **fluid speed** at position 2

$$u_2^2 = u_1^2 + 2\left(\frac{p_1}{\rho} + g\,z_1 - \frac{p_2}{\rho} - g\,z_2\right) \tag{3.58}$$

$$= u_1^2 + 2g\,(z_1 - z_2)\left(1 - \frac{\rho_{\rm atm}}{\rho}\right)\ . \tag{3.59}$$

Bernoulli's equation thus shows how the fluid speed increases with the distance $(z_1 - z_2)$ from the tap. Because the mass density of air is much less than that of water, i.e., $\rho_{\mathrm{atm}}/\rho \approx 10^{-3} \ll 1$, it is a good approximation to write

$$u_2^2 = u_1^2 + 2g\,(z_1 - z_2)\;. \tag{3.60}$$

The **diameter** of the stream of water can be found by applying **mass conservation** to the steady flow between positions 1 and 2: the mass flux of fluid through a circular cross section at z_1 must equal that at z_2:

$$\rho\, u_2\, A_2 = \rho\, u_1\, A_1 = \rho\, u_2 \left(\frac{\pi d_2^2}{4}\right) = \rho\, u_1 \left(\frac{\pi d_1^2}{4}\right)\;. \tag{3.61}$$

Therefore,

$$\begin{aligned} d_2 &= d_1 \sqrt{\frac{u_1}{u_2}} \\ &= d_1 \left(\frac{u_1^2}{u_1^2 + 2g\,(z_1 - z_2)}\right)^{1/4}\;. \end{aligned} \tag{3.62}$$

The diameter only varies slowly with increasing distance from the tap.

3.2.5 Vorticity equation

In (3.50) we already wrote Euler's momentum equation for an incompressible inviscid flow in the form

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{u} = -\,\boldsymbol{\nabla} H\;. \tag{3.63}$$

Now taking the curl of this equation gives (as the order of $\boldsymbol{\nabla}\times$ and $\partial/\partial t$ does not matter in Euclidian space, $\mathbb{R}^3$)

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{\nabla} \times (\boldsymbol{\omega} \times \boldsymbol{u}) = \boldsymbol{0}\;. \tag{3.64}$$

Then, when we apply the vector identity (2.29), we get

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\boldsymbol{u} + \underbrace{\boldsymbol{\omega}\,(\boldsymbol{\nabla}\cdot\boldsymbol{u})}_{\boldsymbol{\nabla}\cdot\boldsymbol{u}=0} - \underbrace{\boldsymbol{u}\,(\boldsymbol{\nabla}\cdot\boldsymbol{\omega})}_{\text{div curl}\,\equiv\,0} = \boldsymbol{0}\,, \tag{3.65}$$

i.e.,

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\boldsymbol{u}\,, \tag{3.66}$$

or, employing the convective derivative defined in (3.5) (and so the **Lagrangian description**),

Vorticity equation:

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\boldsymbol{u}\;. \tag{3.67}$$

As we will see later, this is equation is very useful. It is worth noting that it is a relationship only between $\boldsymbol{u}$ and $\boldsymbol{\omega}$ (i.e., $\boldsymbol{\nabla} \times \boldsymbol{u}$); the pressure has been eliminated.

In the case of a **2-D flow**, where $\boldsymbol{u} = (u_x(t,x,y), u_y(t,x,y), 0)^T$ in a **Cartesian coordinate system**, it follows that

$$\boldsymbol{\omega} = (0, 0, \omega_z)^T \; . \tag{3.68}$$

Then the right-hand side of the vorticity equation becomes

$$(\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\boldsymbol{u} = \omega_z \frac{\partial \boldsymbol{u}}{\partial z} = \mathbf{0} \; , \tag{3.69}$$

and so it follows that

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = \mathbf{0} \; . \tag{3.70}$$

In other words:

> *For two-dimensional flow of an incompressible ideal fluid, subject to a conservative body force, the vorticity $\boldsymbol{\omega}$ of each individual fluid element is conserved.*

We can go further. When a 2-D flow is **steady**, (3.67) reduces to

$$(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{\omega} = \mathbf{0} \; , \tag{3.71}$$

or, in other words:

> *For two-dimensional, steady, flow of an incompressible ideal fluid, subject to a conservative body force, the vorticity $\boldsymbol{\omega}$ is constant along a streamline.*

This conclusion is important since it explains why the *assumption* of irrotational flow is so useful, especially in **aerodynamics**. In the steady flow of air over an aerofoil, the flow is approximately 2-D, and, provided that there are no regions of closed streamlines, all the streamlines can be traced back to “spatial infinity” (e.g., $x = -\infty$). In other words, all the streamlines originate in a region of uniform flow, which has zero vorticity. Thus, from the above conclusion, the vorticity will be zero throughout the flow.

3.2.6 Boundary conditions for ideal fluids

The flow of an ideal fluid is inviscid, so that the only force that a boundary can communicate to the fluid is along the **normal direction** to the boundary surface given by the **unit vector** $\boldsymbol{n}$. Now if the boundary is impermeable (does not let any fluid through), then the boundary forces must prevent any such flow *normal to the surface*. Hence, an ideal fluid "slips" over a boundary without any resistance. It is *this effect* that leads to the strange way of describing an ideal fluid as "dry water". In the present case we speak of a "**flow slip**" boundary condition.

> *For an ideal fluid, the flow velocity at a boundary is tangential to that boundary, and the normal component of the flow velocity at the boundary must equal the normal component of the velocity of the boundary itself.*

If the boundary is at rest, then $\boldsymbol{u} \cdot \boldsymbol{n} = 0$; if the boundary moves at velocity $\boldsymbol{v}$, then $(\boldsymbol{u} - \boldsymbol{v}) \cdot \boldsymbol{n} = 0$. These are called the **kinematic boundary conditions**. It follows that streamlines are *tangent* to any boundary surfaces.

3.3 Viscous fluids

In reality all fluids do have **viscosity**. (Water *is* wet!) We have already introduced viscosity as the **tangential stress** (i.e., force per unit area) that resists any shearing of the fluid velocity. However, not all fluids have the same *kind* of viscosity; their *chemical* and *physical properties* mean they react differently to shear.

Newtonian fluids have a viscosity that is *linearly related* (directly proportional) to the velocity shear, so that for a shear flow $\boldsymbol{u} = (u_x(y), 0, 0)^T$ in a **Cartesian coordinate system** the **shear stress** τ (which acts in the x-direction) is

$$\tau = \mu \frac{\mathrm{d}u_x}{\mathrm{d}y} \; ; \tag{3.72}$$

μ is the **coefficient of shear viscosity**. We also introduce the quantity

$$\nu := \frac{\mu}{\rho} \, , \tag{3.73}$$

which is known as the **coefficient of kinematic viscosity**.

Non-Newtonian fluids show a *different*, typically non-linear, behaviour. For example, some have a viscosity which gets smaller when the shear is very large (e.g., non-drip paint).

3.3.1 Boundary conditions for viscous fluids

Observations of viscous flow around obstacles reveal that the "flow slip" boundary condition of an inviscid fluid (i.e., that the flow velocity is everywhere tangential to a boundary surface) is *not* applicable. In fact, for viscous fluids the flow velocity is actually *zero* on the boundary.

At a rigid boundary both the normal and tangential components of the flow velocity are equal to those of the boundary itself. If the boundary is not moving, then the flow velocity is zero.

This is called the "**no slip**" boundary condition, and holds for any fluid with viscosity, however small that viscosity may be.

3.3.2 Boundary layers

In a **viscous fluid**, where there is some flow, it is observed that there is a thin region adjacent to an obstacle where the flow velocity changes *smoothly, but rapidly*, to zero (not just its normal component). This **boundary layer** arises because the **velocity gradient** is *highest* right next to the obstacle, and, therefore, viscous stresses will be most important. Further from the obstacle, the **velocity gradients** are *smaller* (determined largely by the size of a system/obstacle), and so viscosity plays a smaller role; in those regions the flow can be approximated by assuming it is inviscid.

The existence of boundary layers might be thought a convenient way to treat all flows in the inviscid limit. But a boundary layer is a region of fluid with **shear** and as such feels forces just as other regions of the fluid. And if the pressure forces are in the "wrong direction", it is possible for the boundary layer to *separate* from the obstacle. This leads to behaviour completely different from inviscid flow, and in particular to the formation of a **wake** behind an obstacle in a flow. The wake consists of **turbulent eddies** of streamlines that close on themselves, so that the usual assumption of irrotational flow throughout the fluid is also inapplicable.

If the above seems too negative for the ideal fluid approach, then it should be said that the latter *does* have notable successes. A particularly prominent example is the **lift** produced by steady flow over an aerofoil. But even here it could be noted that one of the main goals of an aircraft wing designer is to *minimize* the wake and the turbulent flow near the ends produced by the wing; these features can never be completely prevented.

3.3.3 Navier–Stokes equations of motion

Here we state the equations of motion governing the flow of an incompressible viscous fluid of the Newtonian kind with constant mass density ρ and constant shear viscosity μ (kinematic viscosity ν); we will be deriving these more generally later in chapter 6. We have (in the **Lagrangian description**)

Navier–Stokes equations:

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} - \nu\,(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\boldsymbol{u} = -\frac{1}{\rho}\,\boldsymbol{\nabla}p + \boldsymbol{g} \tag{3.74}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0\,. \tag{3.75}$$

These equations of motion were first obtained in 1822 by the French physicist Claude Louis Marie Henri Navier (1785–1836), and later rederived independently in 1845 by the Irish mathematician and physicist George Gabriel Stokes (1819–1903). Note that the *difference* between the **Navier–Stokes equations** and **Euler's equations** for an incompressible ideal fluid is the *second-order* derivative term $\nu\,(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\boldsymbol{u}$.

3.3.4 Reynolds number

The **Navier–Stokes momentum equations** (3.74) have a number of terms, and we can characterize the relative importance of the so-called **inertial** and **viscous terms**, $(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}$ and $\nu(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\boldsymbol{u}$, by introducing the *dimensionless number*

$$R := \frac{UL}{\nu}\,. \tag{3.76}$$

This is called the **Reynolds number** after the Irish mathematician and physicist Osborne Reynolds (1842–1912). Here U is a typical flow speed, and L is a typical length scale in the flow over which the fluid quantities vary. The crucial term here is “typical”, by which we mean an order of magnitude estimate only, so that the ratio U/L gives a measure of the **velocity gradients**.

Then we get the estimate:

$$\frac{|\,\textbf{inertial term}\,|}{|\,\textbf{viscous term}\,|} = \frac{O(U^2/L)}{O(\nu U/L^2)} = O(R)\ . \tag{3.77}$$

In other words, when the **Reynolds number** is *large*, the inertial term (the convective, sloshing nature of the fluid) is dominant; when the **Reynolds number** is *small*, the viscous, treacly aspects of the fluid are dominant.

In **high Reynolds number flows** viscous effects should be mostly negligible. The exception is at a boundary layer, where, by definition, the length scale is such that the **Reynolds number** *as calculated for the boundary layer* is no longer large, and viscous effects are important. A high **Reynolds number** is *necessary* for inviscid theory to be valid, but further conditions must be met, e.g.: no separation of boundary layers, no turbulent flow or flow instabilities.

In **low Reynolds number flows** viscous effects are dominant, the fluid's inertia is negligible, and the fluid will stop flowing almost immediately after being given an impulse. There will be no signs of turbulence in such flows, and obviously no distinction between a boundary layer and the main flow region.

3.3.5 Example: steady viscous flow between fixed parallel plates

Several examples of incompressible viscous flow will be given later in chapter 7, after the formal derivation of the **Navier–Stokes equations** in chapter 6. But one straightforward example will be given here.

One-dimensional flow

This is a particular situation where a number of different problems can be solved. In **Cartesian coordinates** one takes the flow velocity, e.g., as $\boldsymbol{u} = (u_x, 0, 0)^T$. The incompressibility condition $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$ implies $\partial u_x/\partial x = 0$, and, hence, the inertial term in the **Navier–Stokes momentum equations**, $(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u}$, is zero (so that the non-linearity drops out of the equations). In the absence of body forces the **Navier–Stokes momentum equations** can then be written as the *linear* partial differential equations:

$$\frac{\partial u_x}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right) \tag{3.78}$$

$$\frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0\ . \tag{3.79}$$

The latter implies that p is independent of y and z. Rearranging the former, we get

$$\frac{\partial p}{\partial x} = \mu\left(\frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right) - \rho\,\frac{\partial u_x}{\partial t}\ . \tag{3.80}$$

Note that the right-hand side *only* contains the velocity component u_x as dependent variable. Now consider again the incompressibility condition:

$$\nabla \cdot \boldsymbol{u} = \frac{\partial u_x}{\partial x} = 0 \; . \tag{3.81}$$

Thus, u_x must be independent of x, and it follows that the right-hand side of (3.80) is also independent of x.

On the other hand, p, and hence the left-hand side $\partial p / \partial x$ of (3.80), in addition *cannot* depend on y and z. Thus, it follows that the only remaining allowed dependence is on t:

$$\frac{\partial p}{\partial x} = - \, G(t) \; , \tag{3.82}$$

where $G(t)$ is some function of time.

Steady flow between fixed parallel plates

We now consider flow in the x–direction between plates parallel to the x–z plane (i.e., planes with $y =$ constant), and, by symmetry, we assume that there also is no variation in the z–direction.

For **steady flow** we must have that the function $G(t)$ is just a constant, k:

$$\frac{\partial p}{\partial x} = \mu \, \frac{\partial^2 u_x}{\partial y^2} = - \, k \; . \tag{3.83}$$

In other words, there is a constant pressure gradient which drives the flow. For the fluid pressure itself we thus get

$$p = p_0 - kx \; , \tag{3.84}$$

while the x–component of the flow velocity is

$$u_x = D + Cy - \frac{ky^2}{2\mu} \; , \tag{3.85}$$

with D and C constants.

Consider further parallel plates at elevations $y = +\,h$ and $y = -\,h$. The boundary condition for viscous fluid is the so-called "**no-slip**" condition, i.e., $u_x = 0$ at $y = \pm\,h$. Thus we can solve for the constants of integration, yielding:

$$C = 0, \qquad D = \frac{kh^2}{2\mu} \; . \tag{3.86}$$

Hence, the final solution is given by

$$\boldsymbol{u} = (u_x, 0, 0)^T \; , \qquad u_x(y) = \frac{k}{2\mu}(h^2 - y^2) \; . \tag{3.87}$$

Note that the flow velocity is proportional to the amplitude of the pressure gradient, k, and the profile of the of flow perpendicular to the flow is *parabolic*.

Chapter 4
Analysis and classification of fluid motion

4.1 Analysis of fluid motion

The aim is to classify the various types of fluid flow according to the geometrical distortion that they imply for any pattern of tracer particles introduced into the flow. If we consider the evolution of a **dyed blob of fluid**, it will not only be swept along with the flow, but will also suffer **expansion (compression)**, **shear** and **rotation** due to the **differential velocities** across the blob.

We begin by examining the **relative motion** of fluid near a *point*, since we are not concerned at the moment with the mean flow. Suppose there is a short **fluid line element** $\mathrm{d}\boldsymbol{l}$ with ends initially at $\boldsymbol{r}_0$ (the reference position) and $\boldsymbol{r}_0 + \mathrm{d}\boldsymbol{l}$. Then the rate of change of this line element, whose ends move with the fluid, is

$$\frac{\mathrm{D}\mathrm{d}\boldsymbol{l}}{\mathrm{D}t} = \boldsymbol{u}(\boldsymbol{r}_0 + \mathrm{d}\boldsymbol{l}) - \boldsymbol{u}(\boldsymbol{r}_0)\ , \tag{4.1}$$

which is just the velocity difference $(\mathrm{d}\boldsymbol{l} \cdot \boldsymbol{\nabla})\boldsymbol{u}$ between the ends of the line element. Note that the ends of the line element are specified using **Eulerian coordinates**.

Therefore, in **Cartesian coordinates**,

$$\left(\frac{\mathrm{D}\mathrm{d}\boldsymbol{l}}{\mathrm{D}t}\right)_x = \mathrm{d}l_x \frac{\partial u_x}{\partial x} + \mathrm{d}l_y \frac{\partial u_x}{\partial y} + \mathrm{d}l_z \frac{\partial u_x}{\partial y}\ , \tag{4.2}$$

which says that the rate of change in time of the x-component of $\mathrm{d}\boldsymbol{l}$ depends on the change of u_x, and this may change with each of the x, y or z coordinates.

We can thus write down a matrix relating the vector $\mathrm{D}\mathrm{d}\boldsymbol{l}/\mathrm{D}t$ with $\mathrm{d}\boldsymbol{l}$,

$$\frac{\mathrm{D}\mathrm{d}\boldsymbol{l}}{\mathrm{D}t} = \begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{pmatrix} \mathrm{d}\boldsymbol{l}\ . \tag{4.3}$$

In index notation this matrix it is the transpose of the matrix with elements

$$u_{ij} = \frac{\partial u_j}{\partial x_i} , \qquad i, j \in \{x, y, z\} , \tag{4.4}$$

which can be split into a sum of a **symmetric matrix** and an **anti-symmetric matrix**:

$$u_{ij} = \underbrace{\frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)}_{\text{symmetric: } \theta_{ij}} + \underbrace{\frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right)}_{\text{anti-symmetric: } \omega_{ij}} . \tag{4.5}$$

4.1.1 Vorticity

We consider first the anti-symmetric matrix. Let us write it in the form

$$\omega_{ij} = \begin{pmatrix} 0 & \omega_{12} & -\omega_{31} \\ -\omega_{12} & 0 & \omega_{23} \\ \omega_{31} & -\omega_{23} & 0 \end{pmatrix} , \tag{4.6}$$

so that in Cartesian coordinates $\omega_{12} = \frac{1}{2}(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y})$.

We have already seen (at least in 2-D) that an anti-symmetric matrix corresponds to a **rotation**. It turns out that operating on $\mathrm{d}\boldsymbol{l}$ with ω_{ij} is equivalent to taking the cross product of $\mathrm{d}\boldsymbol{l}$ with a vector composed of the elements of ω_{ij}, i.e.,

$$\frac{\mathrm{D}\mathrm{d}l_i}{\mathrm{D}t} = \sum_j \omega_{ij}\mathrm{d}l_j \qquad \Rightarrow \qquad \frac{\mathrm{D}\mathrm{d}\boldsymbol{l}}{\mathrm{D}t} = -\left(\omega_{23},\, \omega_{31},\, \omega_{12}\right) \times \mathrm{d}\boldsymbol{l} . \tag{4.7}$$

But from the definition of the elements of ω_{ij} we see that this vector is just $\frac{1}{2}\boldsymbol{\nabla} \times \boldsymbol{u}$. Remember that we have defined the **vorticity** as $\boldsymbol{\omega} := \boldsymbol{\nabla} \times \boldsymbol{u}$, so that the effect of ω_{ij} on $\mathrm{d}\boldsymbol{l}$ is just solid body rotation at a rate $\frac{1}{2}|\boldsymbol{\omega}|$. Hence, we can alternatively write $\boldsymbol{\omega} = 2(\omega_{23}, \omega_{31}, \omega_{12})$.

To emphasize that vorticity is a measure of **local spin**, rather than global rotation, consider a 2-D azimuthal flow (i.e., circular streamlines) with

$$u_\varphi = \frac{u_0}{r} , \tag{4.8}$$

where u_0 is a constant. Applying the curl operator in cylindrical polar coordinates given in (2.14), we find that $\boldsymbol{\nabla} \times \boldsymbol{u} = \boldsymbol{0}$, i.e., the flow is irrotational, even though the fluid is clearly circulating around the z-axis.

In a given fluid region the **circulation** of the flow velocity $\boldsymbol{u}$ around a closed curve ∂S with infinitesimal tangent vector $\mathrm{d}\boldsymbol{l}$ is defined as the line integral [cf. (1.25)]

$$\Gamma := \oint_{\partial S} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{l} . \tag{4.9}$$

With Stokes' integral theorem (2.33) we could write this as

$$\Gamma = \int\!\!\int_S (\boldsymbol{\nabla} \times \boldsymbol{u}) \cdot \mathrm{d}\boldsymbol{A} = \int\!\!\int_S \boldsymbol{\omega} \cdot \mathrm{d}\boldsymbol{A} , \tag{4.10}$$

where S is any surface entirely in the fluid that is spanned by ∂S. This might be interpreted as meaning that flow which is irrotational ($\boldsymbol{\omega} = \mathbf{0}$) will always result in zero circulation. (Think about the flow above with $u_\varphi \propto 1/r$.) But here *care must be taken* that there are *no* obstructions which prevent the surface S being *entirely* in the flow region. The surface S must be **simply connected**.

For example, the $1/r$ rotation implies a **singularity** at the origin, therefore *the origin must be excluded* from the flow domain, so that there is an inner boundary, e.g., at $r = a$. Thus the circulation around a circular path around the origin can be non-zero, even though the flow is irrotational! The same argument applies to 2-D irrotational flow around a wing (where the circulation is intimately associated with the **lift force** felt by the wing).

So, we have an important result:

> *When there are obstacles in the flow, then the circulation around the obstacle can be non-zero, even when the vorticity is everywhere zero.*

If we wish to use Stokes' integral theorem (2.33) when there *are* obstructions, then we must consider paths that *exclude* the obstructions.

In this case

$$\Gamma = u_\varphi(R)2\pi R - u_\varphi(a)2\pi a = 0\ , \qquad (4.11)$$

if $u_\varphi \propto 1/r$, as expected.

4.1.2 Rate of strain tensor

We now return to (4.5) to consider the symmetric part of u_{ij}, i.e., θ_{ij}, which is known as the **rate of strain tensor**.

We have seen, at least in 2-D, that matrices representing pure **shear** have zero trace. Therefore we can make θ_{ij} trace-free by subtracting (with respect to a Cartesian coordinate basis of $\mathbb{R}^3$)

$$\frac{1}{3}\,\theta\,\boldsymbol{g} = \frac{1}{3}\,\mathrm{Tr}(\theta_{ij}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\ . \qquad (4.12)$$

In this expression $\boldsymbol{g}$ denotes the matrix with components $g_{ij} := \boldsymbol{e}_i \cdot \boldsymbol{e}_j$, $i, j = 1, 2, 3$, formed from the scalar products of a set of coordinate basis vectors $\{\,\boldsymbol{e}_1,\, \boldsymbol{e}_2,\, \boldsymbol{e}_3\,\}$ of $\mathbb{R}^3$ (thus, for a Cartesian coordinate basis we just have $g_{ij} = \delta_{ij}$). Moreover, from the definition of the elements of θ_{ij} we see that the **expansion (compression)** scalar θ is

$$\theta = \mathrm{Tr}(\theta_{ij}) = \boldsymbol{\nabla} \cdot \boldsymbol{u} \;, \tag{4.13}$$

so that we can write

$$\theta_{ij} := \sigma_{ij} + \frac{1}{3}\,\theta\, g_{ij} \;, \tag{4.14}$$

where we have decomposed the rate of strain tensor into

- σ_{ij}, the **shear tensor** — symmetric and trace-free, and
- $\frac{1}{3}\,\theta\, g_{ij}$, the **dilatation matrix** — diagonal.

As might be expected from our 2-D example, the dilatation matrix is related to the **volume change** (expansion or compression) in the flow. Consider a **Lagrangian fluid element** (one that moves with the fluid), of volume δV. The total mass $\rho \delta V$ within it will remain constant as it moves along with the fluid, so

$$\frac{\mathrm{D}(\rho \delta V)}{\mathrm{D}t} = 0 \;, \tag{4.15}$$

hence

$$\delta V\, \frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\, \frac{\mathrm{D}\delta V}{\mathrm{D}t} = 0 \;. \tag{4.16}$$

But recall that we have the mass conservation equation

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0 \;, \tag{4.17}$$

or, equivalently,

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\,\rho\, \boldsymbol{\nabla} \cdot \boldsymbol{u} \;. \tag{4.18}$$

Thus it follows that

$$\frac{\mathrm{D}(\delta V)}{\mathrm{D}t} = (\boldsymbol{\nabla} \cdot \boldsymbol{u})\, \delta V = \theta\, \delta V \;. \tag{4.19}$$

In other words, the volume strain (rate of change of volume of a fluid element moving with the fluid, per unit volume) equals the divergence of the flow velocity. The obvious implication is that in an incompressible flow $\theta = \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$. We have already come to this conclusion by a more direct route.

We now finish with the analysis of possible fluid motions by examining the effect of the rate of strain tensor θ_{ij} on the vector $\mathrm{d}\boldsymbol{l}$. The result is a vector whose i-th component can be written as $\sum_j \theta_{ij} \mathrm{d}l_j$.

Taking the curl of this vector, $\nabla \times (\theta_{ij}\mathrm{d}l_j)$, we obtain in index notation (and employing the Einstein summation convention that we sum over repeated indices) that

$$\begin{aligned}
\epsilon_{ijk}\,\partial_j(\theta_{kl}\mathrm{d}l_l) &= \frac{1}{2}\,\epsilon_{ijk}\,\partial_j[\,(\partial_l u_k + \partial_k u_l)\,\mathrm{d}l_l\,] \\
&= \frac{1}{2}\,\epsilon_{ijk}\,[\,\partial_j\partial_l u_k\,\mathrm{d}l_l + \partial_j\partial_k u_l\,\mathrm{d}l_l + 2\,\theta_{kl}\,\delta_{jl}\,] \\
&= \frac{1}{2}\,\mathrm{d}l_l\,\partial_l(\epsilon_{ijk}\partial_j u_k) \\
&= \frac{1}{2}\,\mathrm{d}l_l\,\partial_l\omega_i\ ,
\end{aligned} \tag{4.20}$$

where ϵ_{ijk} denotes the totally anti-symmetric symbol for which $\epsilon_{123} = -\,\epsilon_{213} = 1$ and cyclic permutations thereof, and zero otherwise. The relation we derived means that the curl of the vector $\theta_{ij}\mathrm{d}l_j$ *only* vanishes when the vorticity is independent of position or zero altogether. Note that in deriving this result we used the fact that partial derivatives commute ($\partial_i\partial_j = \partial_j\partial_i$), and that, as the line element $\mathrm{d}l_i$ is just a vector of position coordinates, $\partial_i\mathrm{d}l_j = \delta_{ij}$. Thus, e.g.,

$$\epsilon_{ijk}\,\theta_{jl}\,\delta_{kl} = \epsilon_{ijk}\,\theta_{jk} = 0\ , \tag{4.21}$$

because θ_{jk} is symmetric, but ϵ_{ijk} is totally anti-symmetric, so when summing over two of the indices the result left is a zero i-component.

When the vorticity is position-independent or zero, it follows that the new vector $\theta_{ij}\mathrm{d}l_j$ can be written as the gradient of some scalar potential function ϕ, which turns out to be (remember the summation convention):

$$\phi = \frac{1}{2}\theta_{ij}\mathrm{d}l_i\mathrm{d}l_j\ . \tag{4.22}$$

4.1.3 Summary

This analysis has put on a firm footing the concept that we can characterize different flows according to whether they are **irrotational** (curl-free) and/or **solenoidal** (divergence-free). From the above analysis it is clear that these categories of flow are not arbitrary, but reflect basic distinctions between different kinds of distortions of the fluid flow.

Finally, when the vorticity of a flow is zero, we can write the vector equation for the rate of change of the fluid line element as

$$\frac{\mathrm{D}\mathrm{d}\boldsymbol{l}}{\mathrm{D}t} = \nabla\phi\ ; \tag{4.23}$$

in this case the potential gradient term with $\phi = \frac{1}{2}\theta_{ij}\mathrm{d}l_i\mathrm{d}l_j$ contains the effects of *both* expansion (compression) and shear.

4.2 Classification of fluid flows

4.2.1 Irrotational flows

An irrotational flow satisfies $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u} = \boldsymbol{0}$, so that by the vector analytical identity (2.25) we can express the fluid velocity as the gradient of a scalar potential:

$$\boldsymbol{u} = \boldsymbol{\nabla}\phi \; . \tag{4.24}$$

The **velocity scalar potential** ϕ at some point P is *defined* by

$$\phi := \int_O^P \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{l} \; , \tag{4.25}$$

where O is some fixed reference point. This turns out to be very useful, since it is often easier to solve for a scalar quantity everywhere than a vector quantity. The only problem is that the velocity scalar potential is **single-valued** only if there are *no* obstructions in the flow. In a **simply connected** fluid domain the scalar potential ϕ is independent of the path between O and P, and is thus single-valued. In a multiply connected fluid region (e.g., a domain with an obstruction), ϕ may depend on the path chosen from O to P so that it will be a **multi-valued** function of position. It can be noted that the change in ϕ caused by looping the path around an **obstruction** is just equal to the **circulation** around that obstacle. This can be seen as follows: The circulation around any closed curve C in the flow is

$$\Gamma = \oint_C \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{l} = \oint_C \boldsymbol{\nabla}\phi \cdot \mathrm{d}\boldsymbol{l} = [\phi]_C \; , \tag{4.26}$$

where $[\phi]_C$ denotes the change, if any, in the value in ϕ after one circuit around C.

Velocity scalar potential: examples

- **Uniform flow**

  $$\boldsymbol{u} = (U, 0, 0)^T \; , \qquad U = \text{const} \; ,$$

  is clearly irrotational (and solenoidal) and the velocity scalar potential is

  $$\phi = Ux + k \; , \tag{4.27}$$

  where k is an arbitrary integration constant, which does not affect the flow velocity.

- **Stagnation point flow**

  $$\boldsymbol{u} = u_0(x, -y, 0)^T \; , \qquad u_0 = \text{const} \; ,$$

  is also clearly irrotational (and solenoidal), so we have

  $$\frac{\partial\phi}{\partial x} = u_0 x \; , \qquad \frac{\partial\phi}{\partial y} = -\,u_0\, y \; , \qquad \frac{\partial\phi}{\partial z} = 0 \; . \tag{4.28}$$

Integrating these one finds

$$\phi = \frac{1}{2}u_0(x^2 - y^2) + k \; , \qquad (4.29)$$

where k is an arbitrary integration constant. In this case ϕ is a single-valued function of position.

- **Line vortex flow**

$$\boldsymbol{u}(r, \varphi, z) = (0, u_0/r, 0)^T \; , \qquad u_0 = \text{const} \; , \qquad (4.30)$$

which is irrotational (and solenoidal), except at the origin where it is not defined. As discussed above, the origin has to be *excluded* from the flow domain. This means that the region is not simply connected since closed curves which enclose the origin cannot be shrunk to a point without leaving the flow domain. We integrate:

$$\frac{\partial \phi}{\partial r} = 0 \; , \qquad \frac{1}{r}\frac{\partial \phi}{\partial \varphi} = \frac{u_0}{r} \; , \qquad \frac{\partial \phi}{\partial z} = 0 \; , \qquad (4.31)$$

so that

$$\phi = u_0 \, \varphi \; , \qquad (4.32)$$

which is a multi-valued function of position. Any circuit which goes once around the origin at distance $r = a$ causes φ to increase by 2π, and hence the circulation round such a circuit will be $\Gamma = 2\pi \, u_0$. Thus *all* circuits which go once around the origin will have the *same* circulation.

4.2.2 Incompressible flows

Incompressible flows have $\theta = \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$, and so by the vector analytical identity (2.26) we can write in this case

$$\boldsymbol{u} = \boldsymbol{\nabla} \times \boldsymbol{B} \; , \qquad (4.33)$$

where $\boldsymbol{B}$ is the **velocity vector potential**, in analogy to the case of the velocity scalar potential.

2-D incompressible flows

In some geometries the velocity vector potential $\boldsymbol{B}$ has only one component, and is thus particularly easy to deal with.

Consider a 2-D flow, in **Cartesian coordinates**, such that $\partial \boldsymbol{u}/\partial z = 0$. Assume we can write $\boldsymbol{B} = (0, 0, B_z)^T$ where

$$\frac{\partial B_z}{\partial y} = u_x \; , \qquad \frac{\partial B_z}{\partial x} = -\, u_y \; . \qquad (4.34)$$

One can check that this agrees with $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$:

$$\frac{\partial^2 B_z}{\partial x \partial y} = \frac{\partial^2 B_z}{\partial y \partial x} \qquad \Rightarrow \qquad \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 \; . \qquad (4.35)$$

To understand the physical interpretation of $\boldsymbol{B}$, consider a section, of volume V, of a **stream tube**, where the section is bounded by two surfaces 1 and 2. (A stream tube is *defined* by translating a closed curve everywhere in the direction of the fluid velocity, so that the component of the velocity normal to the stream tube surface is everywhere zero. The surface of a stream tube thus consists of streamlines passing through the closed curve.)

By Gauß' integral theorem (2.32) applied to incompressible flow, we have

$$\int\int\int_V \boldsymbol{\nabla}\cdot\boldsymbol{u}\,\mathrm{d}V = \int\int_1 \boldsymbol{u}\cdot\mathrm{d}\boldsymbol{A} - \int\int_2 \boldsymbol{u}\cdot\mathrm{d}\boldsymbol{A} = 0\ , \qquad (4.36)$$

since $\boldsymbol{u}$ is necessarily everywhere tangent to the surface of the stream tube. This implies that the **volume flux** $\int\int \boldsymbol{u}\cdot\mathrm{d}\boldsymbol{A}$ is *constant* for *any* surface spanning the stream tube. Then, by Stokes' integral theorem (2.33), $\oint \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{l}$ is *constant* along the edge of any cross section. So, if there are two streamlines (a and b in the figure), then, as $B_x = B_y = 0$, it follows that the volume flux per unit time per unit height in the z-direction is

$$\int\int \boldsymbol{u}\cdot\mathrm{d}\boldsymbol{A} = \oint \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{l} = B_z|_a - B_z|_b\ . \qquad (4.37)$$

Since we have already seen that this volume flux per unit height across the stream tube is the same for any cross section, and equal to $B_z|_a - B_z|_b$, it follows that B_z must be *constant along a streamline*. We can check this from

$$(\boldsymbol{u}\cdot\boldsymbol{\nabla})B_z = u_x\frac{\partial B_z}{\partial x} + u_y\frac{\partial B_z}{\partial y} = 0\ , \qquad (4.38)$$

because $u_x = \partial B_z/\partial y$ and $u_y = -\,\partial B_z/\partial x$.

The function

$$B_z = B_z(x, y)$$

is then what is known as a **stream function**. In the case of 2-D Cartesian flows it is called the **Lagrange stream function** ψ.

4.2.3 Axisymmetric incompressible flows

Similar arguments can be made for the case of axisymmetric incompressible flows, where, for **cylindrical polar coordinates** $\{r, \varphi, z\}$, one has $\partial\boldsymbol{u}/\partial\varphi = \mathbf{0}$. In this case, $\boldsymbol{B} = (0, B_\varphi, 0)^T$, and so the **stream function** is

$$\psi = r\,B_\varphi\ ,$$

with $B_\varphi = B_\varphi(r, z)$. For historical reasons, the function ψ is now called the **Stokes stream function**. It corresponds to the **volume flux** per unit time through a circle with $z = \text{constant}$ taking the value $2\pi\,\psi$.

Chapter 5
Irrotational flows of incompressible fluids

5.1 Irrotational and incompressible flows

5.1.1 Laplace's equation

In this chapter we will concentrate on inviscid incompressible fluids, discussing examples of flows that are irrotational. For an **irrotational flow** the **flow velocity** can be written as the gradient of a **velocity scalar potential** ϕ, defined according to

$$\boldsymbol{u} := \boldsymbol{\nabla}\phi \; ; \tag{5.1}$$

the physical dimension of ϕ is $[\text{length}]^2[\text{time}]^{-1}$. On the other hand, for an **incompressible flow** we have $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$. Therefore, for a flow that is *both* irrotational and incompressible, the **velocity scalar potential** must satisfy the *linear* partial differential equation of second order

$$(\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\phi = 0 \; . \tag{5.2}$$

This is **Laplace's equation**, which is named after the French physicist and mathematician Pierre-Simon Laplace (1749–1827). It has wide applications throughout physics. For example, it provides the equation for determinating (Newtonian) gravitational or electrostatic scalar potentials outside, respectively, isolated static mass or electric charge distributions (cf. the lectures on MAS107 Newtonian Dynamics and Gravitation and MAS207 Electromagnetism). There are different ways to *solve* **Laplace's equation**, and we may choose the method and solution types to match any particular fluid flow situation. Note that for problems of the kind discussed in this chapter we may use **Euler's momentum equation** to solve for the thermodynamical pressure and the **continuity equation** to solve for the mass density, once a solution to (5.2) (and so by (5.1) for $\boldsymbol{u}$) has been obtained.

Since the flow is **incompressible** (as well as irrotational), one can also derive a **velocity vector potential** $\boldsymbol{B}$ and a related **stream function** ψ (cf. chapter 4).[1] The *definition* of a stream function is provided by the condition

$$(\boldsymbol{u} \cdot \boldsymbol{\nabla})\psi := 0 \; . \tag{5.3}$$

[1]For example, given an incompressible flow that is effectively 2-D, in Cartesian coordinates a velocity vector potential and associated stream function are given by $\boldsymbol{B} = \psi(x, y, z)\, \boldsymbol{e}_z$. Then $\boldsymbol{u} = \boldsymbol{\nabla} \times \boldsymbol{B}$, and $(\boldsymbol{u} \cdot \boldsymbol{\nabla})\psi = 0$ is identically satisfied.

Thus with $\boldsymbol{u} = \boldsymbol{\nabla}\phi$ we see that $\boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\psi = 0$. In other words, lines of constant ϕ and lines of constant ψ lie at *right angles* to each other. This is what one would expect, since lines of constant ψ are just streamlines whereas lines of constant ϕ must be perpendicular to streamlines in order that $\boldsymbol{u} = \boldsymbol{\nabla}\phi$ points along streamlines.

5.1.2 Existence and uniqueness theorems

To discuss the **existence theorem** for solutions to **Laplace's equation** is beyond the scope of this course (and most other undergraduate mathematics courses).

On the other hand, you will find the **uniqueness theorem** for smooth solutions to **Laplace's equation** discussed in MAS204 Calculus III. Here we just state the conclusions that can be drawn from its application.

For irrotational and incompressible flow, the **uniqueness** of a smooth **flow velocity** vector field $\boldsymbol{u} = \boldsymbol{\nabla}\phi$ is guaranteed provided that either the **velocity scalar potential** ϕ or the **normal component** of the flow velocity, $\boldsymbol{n} \cdot \boldsymbol{u} = (\boldsymbol{n} \cdot \boldsymbol{\nabla})\phi$, is specified *at the boundary* of the fluid domain of integration.

The **uniqueness theorem** can be generalised for other circumstances and types of flow — we may return to this!

5.2 Sources, sinks and dipoles

We now consider certain simple, standard, solutions of **Laplace's equation** and investigate the flows corresponding to these solutions.

The simplest solution of $(\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\phi = 0$ is $\phi = \text{constant}$ (or $\phi = 0$), which corresponds to the *trivial case* $\boldsymbol{u} = \boldsymbol{0}$.

5.2.1 Sources and sinks

The function

$$\phi = -\frac{q}{r} \, , \tag{5.4}$$

where q is a constant of physical dimension unit volume per unit time, is a **basic solution** of **Laplace's equation**. Here r is the distance from the origin of a chosen coordinate system. To confirm this is a solution indeed, one could work in **Cartesian coordinates** from $r = (x^2 + y^2 + z^2)^{1/2}$, or, more simply, from **spherical polar coordinates** where

$$(\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \dots \, , \tag{5.5}$$

(cf. (2.22) in chapter 2). The corresponding **flow velocity** is then given by

$$\boldsymbol{u} = \boldsymbol{\nabla}\phi = \frac{q}{r^3}\,\boldsymbol{r} \, , \tag{5.6}$$

where $\boldsymbol{r}$ is the **position vector** relative to the origin of the field point at which $\boldsymbol{u}$ is evaluated; in **Cartesian coordinates** $\boldsymbol{r} = (x, y, z)^T$. We see that the **flow velocity** is everywhere *radial* and of *magnitude* q/r^2; it represents steady streaming of fluid away from the origin. One can ask how much mass passes through a sphere, of radius r, centred on the origin, in unit time, i.e., we can compute the **mass flux**. Assuming $\rho =$ constant,

$$\frac{\mathrm{d}m}{\mathrm{d}t} = I_m = \int\int_{\partial G} \rho \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}A = \rho \int\int_{\partial G} \frac{q}{r^2} \, r^2 \, \mathrm{d}\Omega = 4\pi\rho q \; . \qquad (5.7)$$

The differential $\mathrm{d}\Omega$ represents integration over all infinitesimal solid angles on a unit sphere, i.e., $\mathrm{d}\Omega = \sin\vartheta \mathrm{d}\vartheta \mathrm{d}\varphi$. Obviously this mass flux is independent of the radius of the sphere. From the **continuity equation** it follows that the mass must *enter* at the origin and flow radially away. The flow would be approximated by a pipe pumping water into the middle of a sea. The **velocity scalar potential**

$$\phi = -\,\frac{q}{r} = -\,\frac{I_m}{4\pi\rho}\,\frac{1}{r} \; ,$$

thus represents a **source** at the origin whose strength is measured by q (or by I_m).

If the strength q is negative, i.e., fluid *leaves* the region at the origin, then the flow contains a so-called **sink**.

Since **Laplace's equation** is *linear*, the flow pattern for *several* sources/sinks can be found simply by *adding* the potentials for the individual sources/sinks. This is generally referred to as the **superposition principle**.

5.2.2 Line source

One can also have a **line source** when one considers 2-D flows. The line source is characterised by emitting a mass m per unit length. Working in **cylindrical polar coordinates**, one notes that $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$ implies

$$\frac{\partial}{\partial r}\,(r u_r) = 0 \qquad \Rightarrow \qquad u_r \propto \frac{1}{r} \; . \qquad (5.8)$$

To find the **mass flux per unit length**, consider a cylindrical shell at radius r enclosing the line source. One finds $I_m = 2\pi\rho r u_r(r)$, so that

$$u_r = \frac{I_m}{2\pi\rho}\,\frac{1}{r} \; , \qquad (5.9)$$

and the **velocity scalar potential** is

$$\phi = \frac{I_m}{2\pi\rho}\,\ln r + \text{constant} \; . \qquad (5.10)$$

5.2.3 Solid harmonics

Starting from the basic solution of $1/r$, where $r = (x^2 + y^2 + z^2)^{1/2}$, an *important* set of solutions of **Laplace's equation** can be obtained by differentiating $1/r$ with respect to the coordinates. The functions found in this way are homogeneous functions of *negative* degree of the **Cartesian coordinates**.

- Degree -2:

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right), \quad \frac{\partial}{\partial y}\left(\frac{1}{r}\right), \quad \frac{\partial}{\partial z}\left(\frac{1}{r}\right) . \tag{5.11}$$

- Degree $-(n+1)$:

$$\frac{\partial^n}{\partial x^p \partial y^q \partial z^r}\left(\frac{1}{r}\right) , \qquad p+q+r=n . \tag{5.12}$$

These are solutions since $(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})$ and $\partial^n/(\partial x^p\partial y^q\partial z^r$ commute for all values of p, q, r and $n=p+q+r$, and $1/r$ is a basic solution of **Laplace's equation**:

$$(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\frac{\partial^n}{\partial x^p \partial y^q \partial z^r}\left(\frac{1}{r}\right) = \frac{\partial^n}{\partial x^p \partial y^q \partial z^r}(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\left(\frac{1}{r}\right) = 0 . \tag{5.13}$$

Another set of harmonic functions which are homogeneous and of *positive* degree can be obtained by multiplying the functions in the brackets in (5.11) by r^3 and in (5.12) by r^{2n+1}.

Both sets of harmonic functions are referred to as the **solid harmonics**. They form a "complete set" of harmonic functions because an arbitrary solution which is homogeneous of degree k can be found as a linear combination of harmonic functions of degree k (see also chapter 2).

The solutions of negative degree have **singularities** at the origin but tend to zero as $r\to\infty$. Since they correspond to solutions which exclude the origin they are termed **external harmonics**. Solutions of positive degree apply in a region that may include the origin, but does not extend to infinity; they are called **internal harmonics**.

5.2.4 Dipoles

We now consider the **external harmonics of degree** -2. The general **velocity scalar potential** as a solution of **Laplace's equation**, homogeneous of **degree** -2, is made up of a **linear combination** of the functions in (5.11)

$$\begin{aligned}\phi &= \left(\mu_1\frac{\partial}{\partial x}+\mu_2\frac{\partial}{\partial y}+\mu_3\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right)\\ &= -\frac{(\mu_1 x+\mu_2 y+\mu_3 z)}{r^3} = -\frac{\boldsymbol{\mu}\cdot\boldsymbol{r}}{r^3} .\end{aligned} \tag{5.14}$$

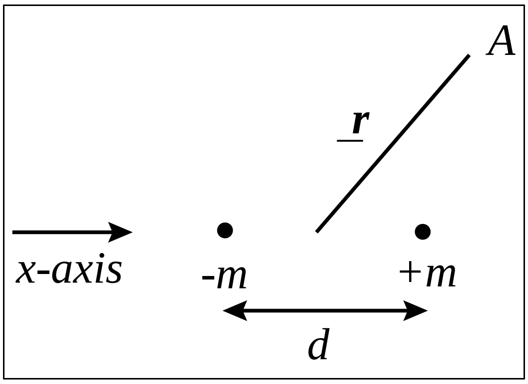
Here $\boldsymbol{\mu}$ is a constant vector of the arbitrary coefficients in the linear combination. An alternative expression for ϕ is

$$\phi = -\frac{\boldsymbol{\mu}\cdot\boldsymbol{r}}{r^3} = (\boldsymbol{\mu}\cdot\boldsymbol{\nabla})\frac{1}{r} . \tag{5.15}$$

The corresponding **flow velocity** is

$$\boldsymbol{u} = \boldsymbol{\nabla}\phi = -\frac{1}{r^3}\boldsymbol{\nabla}(\boldsymbol{\mu}\cdot\boldsymbol{r}) - (\boldsymbol{\mu}\cdot\boldsymbol{r})\boldsymbol{\nabla}\frac{1}{r^3} = -\frac{\boldsymbol{\mu}}{r^3} + \frac{3(\boldsymbol{\mu}\cdot\boldsymbol{r})\boldsymbol{r}}{r^5} . \tag{5.16}$$

Note the fall-off of its magnitude with $1/r^3$ at the highest order in r. This particular solution corresponds to what is called a **dipole** which, in analogy to electrostatics where a dipole is a close pair of equal but opposite electric charges, represents a **source** and a **sink** of equal but opposite strengths close together.



We can demonstrate this by considering a source and a sink (strengths m and $-m$) separated by a distance d in the $\boldsymbol{e}_x$–direction. By the **superposition principle** the total potential at a point A with position vector $\boldsymbol{r}$ is

$$\phi = \phi_{\text{source}} + \phi_{\text{sink}} \ ,$$

hence

$$\phi = \frac{-m}{4\pi\rho\left[(\boldsymbol{r} - (d/2)\,\boldsymbol{e}_x)^2\right]^{1/2}} + \frac{m}{4\pi\rho\left[(\boldsymbol{r} + (d/2)\,\boldsymbol{e}_x)^2\right]^{1/2}} \ . \tag{5.17}$$

Writing out the denominators in terms of scalar products and using the binomial theorem to expand to first order in d/r gives first

$$\phi = \frac{-m}{4\pi\rho}\left\{\left(\boldsymbol{r}\cdot\boldsymbol{r} - d\,\boldsymbol{e}_x\cdot\boldsymbol{r} + \frac{d^2}{4}\right)^{-1/2} - \left(\boldsymbol{r}\cdot\boldsymbol{r} + d\,\boldsymbol{e}_x\cdot\boldsymbol{r} + \frac{d^2}{4}\right)^{-1/2}\right\} , \tag{5.18}$$

then

$$\begin{aligned}\phi &= \frac{-m}{4\pi\rho r}\left\{\left(1 - \frac{d\,\boldsymbol{e}_x\cdot\boldsymbol{r}}{r^2} + \mathcal{O}\left(\frac{d^2}{r^2}\right)\right)^{-1/2}\right.\\ &\qquad \left. - \left(1 + \frac{d\,\boldsymbol{e}_x\cdot\boldsymbol{r}}{r^2} + \mathcal{O}\left(\frac{d^2}{r^2}\right)\right)^{-1/2}\right\} ,\end{aligned} \tag{5.19}$$

and so

$$\phi = \frac{-md\,\boldsymbol{e}_x\cdot\boldsymbol{r}}{4\pi\rho r^3} + \mathcal{O}\left(\frac{d^2}{r^2}\right) = -\frac{\boldsymbol{\mu}\cdot\boldsymbol{r}}{4\pi\rho r^3} + \mathcal{O}\left(\frac{d^2}{r^2}\right) , \tag{5.20}$$

where we have defined $\boldsymbol{\mu} = md\,\boldsymbol{e}_x$ as the **dipole strength**. Note that $\boldsymbol{\mu}$ gives both the strength and orientation of the dipole.

In **spherical polar coordinates**, assuming $\boldsymbol{e}_x$ was aligned with the coordinate line $\vartheta = 0$, the **dipole velocity scalar potential** can be written as

$$\phi = -\frac{\mu \cos\vartheta}{4\pi\rho r^2} \; . \tag{5.21}$$

Dipole streamlines.

An equivalent derivation is obtained by noting that ϕ_{source} and ϕ_{sink} differ because the sign of m is changed and because the sink is displaced by $-\boldsymbol{d}$ with respect to the source. Thus the potential at a point A due to the sink is equal in magnitude to the potential of a **source** at $\boldsymbol{r} + \boldsymbol{d}$. Since $\boldsymbol{d}$ is small we can then use the directional derivative to approximate

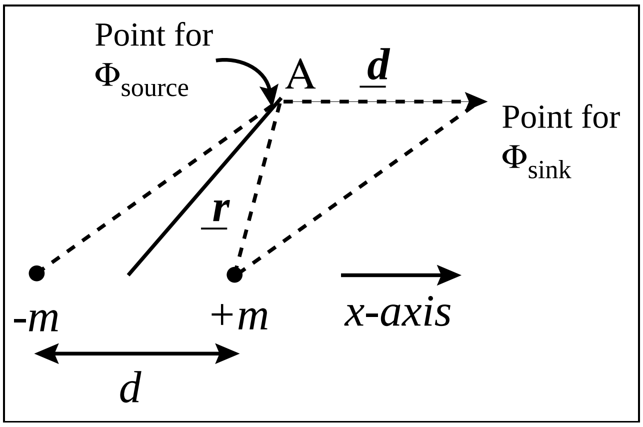
$$\phi_{\text{source}}(\boldsymbol{r} + \boldsymbol{d}) = \phi_{\text{source}}(\boldsymbol{r}) + \boldsymbol{d} \cdot \boldsymbol{\nabla}\phi_{\text{source}}|_{\boldsymbol{r}} \;\; , \tag{5.22}$$

so that

$$\phi_{\text{sink}} = -\left(\phi_{\text{source}} + \boldsymbol{\nabla}\phi_{\text{source}} \cdot d\boldsymbol{e}_x\right) , \tag{5.23}$$

and then the total potential at A is just

$$\phi = \phi_{\text{sink}} + \phi_{\text{source}} = -\boldsymbol{\nabla}\phi_{\text{source}} \cdot d\boldsymbol{e}_x \; . \tag{5.24}$$



5.3 Examples

5.3.1 Sphere moving through fluid at constant velocity

We now look at examples of constructing solutions that describe irrotational and incompressible flows of inviscid fluids, using the basic solutions of the previous paragraph (**sources**, **sinks** and **dipoles**). Here we rely on both the **linearity** of **Laplace's equation**, so that we can add different potentials together to form new solutions, and on the **uniqueness theorem**, which means that if we have found a solution that satisfies the **boundary conditions** then we can be assured that the solution is the one and only correct one.

Consider a **sphere** of radius a moving with a steady velocity of magnitude $U =$ constant through an ideal fluid that is at rest "at infinity". We wish to obtain the **velocity scalar potential** and the equations of the **streamlines**.

We use a (moving) coordinate system which instantaneously has its origin at the centre of the sphere so that the region occupied by the fluid *excludes* the origin and extends to infinity. Therefore we assume an **external harmonic** as an *Ansatz* for the **velocity scalar potential**. *On* the sphere the **normal component** of the **flow velocity** must be equal to the normal component of the velocity of the boundary, i.e., if the location of a position on the boundary is specified by an angle ϑ, then

$$u_r = \frac{\partial \phi}{\partial r} = U \cos \vartheta \quad \text{at} \quad r = a \, . \tag{5.25}$$

Using **spherical polar coordinates** with the polar axis parallel to the flow velocity, we try to fit a solution of the form

$$\phi = -b \, \frac{\mu \cos \vartheta}{r^2} \tag{5.26}$$

with an arbitrary constant b to the boundary conditions. We choose this *Ansatz* because it is the *only* external harmonic which depends on ϑ only through the function $\cos \vartheta$, and this is the *same* dependence as in the **boundary condition** (5.25). Thus we obtain

$$\frac{\partial \phi}{\partial r} = b \, \frac{2\mu \cos \vartheta}{r^3} \, , \tag{5.27}$$

so that on the boundary $r = a$

$$b \, \frac{2\mu \cos \vartheta}{a^3} = U \cos \vartheta \qquad \Rightarrow \qquad b = \frac{U a^3}{2\mu} \, . \tag{5.28}$$

Therefore, the solution for the **velocity scalar potential** is

$$\phi = -\frac{U a^3 \cos \vartheta}{2 r^2} \, . \tag{5.29}$$

This implies that the flow produced by the motion of the sphere is that of a **dipole** of strength $Ua^3/2b$ for the region *outside* the sphere.

At any point in the fluid we use $\boldsymbol{u} = \boldsymbol{\nabla} \phi$ to find the components of the **flow velocity**

$$u_r = \frac{\partial \phi}{\partial r} = \frac{U a^3 \cos \vartheta}{r^3} \, , \qquad u_\vartheta = \frac{1}{r} \frac{\partial \phi}{\partial \vartheta} = \frac{U a^3 \sin \vartheta}{2 r^3} \, . \tag{5.30}$$

The azimuthal component u_φ is zero, because by the axial symmetry of the motion of the sphere ("zero swirl") we have $\partial \phi / \partial \varphi = 0$.

The **streamlines** are found from the differential equations

$$\frac{\mathrm{d}r}{(Ua^3/r^3)\cos\vartheta} = \frac{r\mathrm{d}\vartheta}{(Ua^3/2r^3)\sin\vartheta} , \qquad r\sin\vartheta\mathrm{d}\varphi = 0 . \tag{5.31}$$

This leads to

$$\frac{dr}{r} = \frac{2\cos\vartheta\mathrm{d}\vartheta}{\sin\vartheta} , \qquad \mathrm{d}\varphi = 0 , \tag{5.32}$$

and hence

$$r = A\sin^2\vartheta , \qquad \phi = B , \tag{5.33}$$

where A and B are constants which specify a particular streamline. The flow is obviously axially symmetric (it does *not* depend on φ).

It should be noted that the motion is *not* steady. The dipolar pattern of streamlines gives the flow for the instant that the centre of the sphere *coincides* with the origin of the coordinates. At a subsequent instant the whole pattern has moved to a new position as the centre is in motion.

5.3.2 Source in a uniform stream

This example illustrates how different potential solutions can be added together to create a new solution. Consider a **point source** which steadily emits a **volume** Q of liquid per unit time (note that previously we characterised the strength of the source by the **mass** emitted per unit time). This source shall lie at the origin in a three-dimensional **uniform stream** with velocity magnitude U = constant, which is parallel to the z–axis, i.e., $\boldsymbol{u} = U\,\boldsymbol{e}_z$. We will find the **streamlines** in this case and show that there is *one* streamline which separates streamlines which start at the point source and streamlines which start at infinity upstream.

We use **spherical polar coordinates**, with the z–axis along the polar axis. The **velocity scalar potential** is simply the sum of the potentials corresponding to the **uniform stream** and the **point source**:

$$\phi = Ur\cos\vartheta - \frac{Q}{4\pi r} . \tag{5.34}$$

From $\boldsymbol{u} = \boldsymbol{\nabla}\phi$ we find

$$u_r = \frac{\partial\phi}{\partial r} = U\cos\vartheta + \frac{Q}{4\pi r^2} , \qquad u_\vartheta = \frac{1}{r}\frac{\partial\phi}{\partial\vartheta} = -\,U\sin\vartheta , \qquad u_\varphi = 0 . \tag{5.35}$$

Since $u_\varphi = 0$, the streamlines lie in azimuthal planes, of which the y–z plane is typical. In this plane the differential equations for the **streamlines** are

$$\frac{\mathrm{d}r}{U\cos\vartheta + Q/(4\pi r^2)} = \frac{r\mathrm{d}\vartheta}{-U\sin\vartheta} \tag{5.36}$$

and

$$\frac{1}{r}\frac{\mathrm{d}r}{\mathrm{d}\vartheta} = -\cot\vartheta - \frac{Q}{4\pi U r^2\sin\vartheta} \,. \tag{5.37}$$

Multiplying by $r^2\sin^2\vartheta$, so that

$$r\sin^2\vartheta\frac{\mathrm{d}r}{\mathrm{d}\vartheta} + r^2\sin\vartheta\cos\vartheta = -\frac{Q\sin\vartheta}{4\pi U} \,, \tag{5.38}$$

and substituting $W = r^2\sin^2\vartheta$, noting that

$$\frac{\mathrm{d}W}{\mathrm{d}\vartheta} = 2r\sin^2\vartheta\frac{\mathrm{d}r}{\mathrm{d}\vartheta} + 2r^2\sin\vartheta\cos\vartheta \,, \tag{5.39}$$

one can integrate this equation to find

$$r^2\sin^2\vartheta = \frac{Q}{2\pi U}\cos\vartheta + A \,, \tag{5.40}$$

where A is a constant which varies from streamline to streamline.

There is a **stagnation point** N where $\boldsymbol{u}|_N = \mathbf{0}$, i.e., at

$$\vartheta = \pi \,, \qquad r^2 = \frac{Q}{4\pi U} \,. \tag{5.41}$$

For the streamline that passes through this stagnation point, the constant A takes the value $A = Q/(2\pi U)$, and the corresponding streamline equation is

$$r^2\sin^2\vartheta = \frac{Q}{2\pi U}(1+\cos\vartheta) \,. \tag{5.42}$$

Writing $\sin^2\vartheta = (1-\cos^2\vartheta)$ and rearranging, we obtain the alternative form

$$(1+\cos\vartheta)\left[r^2(1-\cos\vartheta) - \frac{Q}{2\pi U}\right] = 0 \,, \tag{5.43}$$

which has solutions

$$\vartheta = \pi \,, \qquad r^2 = \frac{Q}{2\pi U(1-\cos\vartheta)} \,. \tag{5.44}$$

From this result it is clear that the stagnation point is actually where two particular streamlines meet: one from the source and one from upstream.

The equation for streamlines can now be written as

$$r^2\sin^2\vartheta = \frac{Q}{2\pi U}(1+\cos\vartheta) + A' \,, \tag{5.45}$$

where $A' = A - Q/(2\pi U)$, so that

$$r^2 = \frac{Q}{2\pi U(1-\cos\vartheta)} + \frac{A'}{\sin^2\vartheta} \,. \tag{5.46}$$

Since away from the z–axis we have $\sin^2\vartheta > 0$: if $A' > 0$, a streamline curve lies *above* the **separatrix curve** (5.42); conversely, if $A' < 0$, a streamline curve lies *below*. Hence, the curve (5.42) divides the streamlines according to whether they originate at the point source at the origin, or at infinity upstream.

Streamlines.

We can go slightly further with this example to calculate the **force** (“drag”) exerted on the source by the uniform stream. Since the flow is steady and irrotational, **Bernoulli’s streamline theorem** states

$$\frac{1}{2}\,U^2 + \frac{QU\cos\vartheta}{2\pi r^2} + \frac{Q^2}{16\pi^2 r^4} + \frac{p}{\rho} = \text{constant}\ . \tag{5.47}$$

Consider a small sphere of radius r centred on the source. We can use the above equation to find the pressure on the sphere. Consider a small surface element of the sphere, $\boldsymbol{n}\mathrm{d}A$. The only force acting across it is the **pressure force**, $-p\boldsymbol{n}\mathrm{d}A$ (the minus sign because $\boldsymbol{n}\mathrm{d}A$ points out from sphere). By symmetry the x– and y–components will sum to zero, so we only consider the z–component of this force, $-\cos\vartheta\,p\boldsymbol{n}\mathrm{d}A$.

Integrating this component over the sphere and noting that only one term in the pressure equation is not symmetric, we find the z–directed force (in the direction of the stream) to be

$$-\frac{\rho QU}{2\pi r^2}\int_0^{2\pi}\int_0^{\pi} r^2\cos^2\vartheta\sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi\;\boldsymbol{e}_z = \frac{2}{3}\,\rho\,QU\;\boldsymbol{e}_z\ . \tag{5.48}$$

5.3.3 Method of images

This is a useful application of the **uniqueness theorem**. In this method a problem consisting of **sources** and **boundaries** can be replaced by an equivalent problem consisting of **sources** *only*. This is accomplished by adding sources until the right conditions are created at the position of the boundaries. From the uniqueness theorem it then follows that the resultant flow must everywhere be the same as in the original problem. For example, a problem with a single source near a plane boundary can be replaced by a problem with two symmetrically placed sources, and this has the necessary property of zero normal velocity at the position of the boundary. Further information on this method and examples can be found in the online script to MAS207 Electromagnetism.

5.4 Solutions of Laplace's equation

5.4.1 Spherical harmonics

We now turn to a slightly more comprehensive discussion of obtaining **particular solutions** of **Laplace's equation**. We will first discuss a special class of solutions which are referred to as **spherical harmonics**.

Here one attacks the problem of obtaining a particular solution of **Laplace's equation** by assuming that solutions to (5.2) exist which are **separable** as regards their coordinate dependencies. Clearly this assumption provides a drastic simplification of matters.

Let us choose **spherical polar coordinates**. Let us then take the **separation of variables Ansatz**

$$\phi(r,\vartheta,\varphi) = R(r)\,P(\vartheta)\,Q(\varphi) \tag{5.49}$$

and substitute it into **Laplace's equation**. By (2.22) we know that in spherical polar coordinates (5.2) reads

$$(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\,\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\,\frac{\partial\phi}{\partial\vartheta}\right)+\frac{1}{r^2\sin^2\vartheta}\frac{\partial^2\phi}{\partial\varphi^2} = 0\,. \tag{5.50}$$

Upon substitution of (5.49), and dividing by ϕ/r^2, we obtain

$$\frac{1}{R}\,\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)+\frac{1}{P\sin\vartheta}\frac{\mathrm{d}}{\mathrm{d}\vartheta}\left(\sin\vartheta\,\frac{\mathrm{d}P}{\mathrm{d}\vartheta}\right)+\frac{1}{Q\sin^2\vartheta}\frac{\mathrm{d}^2Q}{\mathrm{d}\varphi^2} = 0\,. \tag{5.51}$$

Note that partial derivatives are not needed here because of the separate dependencies of the different factors of ϕ in (5.49).

The essence of the **method of separation of variables** is to manipulate the linear partial differential equation (5.2) (after assuming the separated form of a solution) in such a way that it decomposes into *two parts*, where the two parts have *no* independent variable in common. It then follows that the two parts must be *separately constant*. In the above case we see that the equation has indeed separated into two parts, one depending only on the independent variable r and the other depending only on the independent variables ϑ and φ. Hence, we must have **constancy** of these two parts such that one part always cancels the other for any arbitrary values of (r, ϑ, φ), in order to satisfy (5.2). For convenience we choose this constant to be

$$l(l+1) \ ,$$

where l is a positive integer or zero. Then we obtain the linear ordinary differential equation of second order

$$\frac{1}{R}\,\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\,\frac{\mathrm{d}R}{\mathrm{d}r}\right) = l(l+1) \ , \tag{5.52}$$

to determine the radial dependence of (5.49), and

$$l(l+1)\,\sin^2\vartheta + \frac{\sin\vartheta}{P}\,\frac{\mathrm{d}}{\mathrm{d}\vartheta}\left(\sin\vartheta\,\frac{\mathrm{d}P}{\mathrm{d}\vartheta}\right) + \frac{1}{Q}\,\frac{\mathrm{d}^2Q}{\mathrm{d}\varphi^2} = 0 \ . \tag{5.53}$$

The latter differential equation is now again *separable*. We take the arbitrary constant thus arising as

$$m^2 \ ,$$

so that we obtain the linear ordinary differential equation of second order

$$[\,l(l+1)\,\sin^2\vartheta - m^2\,]\,P + \sin\vartheta\,\frac{\mathrm{d}}{\mathrm{d}\vartheta}\left(\sin\vartheta\,\frac{\mathrm{d}P}{\mathrm{d}\vartheta}\right) = 0 \ , \tag{5.54}$$

which determines the polar dependence of (5.49), and the linear ordinary differential equation of second order

$$\frac{\mathrm{d}^2Q}{\mathrm{d}\varphi^2} + m^2\,Q = 0 \ , \tag{5.55}$$

which determines the azimuthal dependence of (5.49).

Now the r–dependent equation (5.52) is easily solved by

$$R(r) = A\,r^l + B\,r^{-(l+1)} \ , \tag{5.56}$$

where A and B are arbitrary integration constants. On the other hand, the φ–dependent equation (5.55) has the straightforward solution

$$Q(\varphi) = C\,\cos m\varphi + D\,\sin m\varphi \ , \tag{5.57}$$

where again C and D are arbitrary integration constants. For $Q(\varphi)$ to be *singled-valued* one must have $Q(\varphi + 2\pi) = Q(\varphi)$, and thus it follows that m must be *integral*. Without loss of generality we assume that it can either be zero or a positive integer.

Solving the ϑ–dependent equation (5.54) is far more tricky. Full details on this are given in the online script to MAS207 Electromagnetism. Once this is accomplished, the **spherical harmonics** as solutions of a **2-D Laplace's equation** on the surface of a **unit sphere** (with unit radius) will be given by the product $Y_{lm}(\vartheta, \varphi) \propto P(\vartheta)\,Q(\varphi)$, appropriately normalised. For reasons of complexity we will confine ourselves in the following to a simple subcase of the differential equation (5.54) and, thus, (5.50).

5.4.2 Axisymmetric case: Legendre's equation

By definition a fluid configuration is **axisymmetric** when for the **velocity scalar potential** the property

$$\frac{\partial \phi}{\partial \varphi} = 0$$

holds (which corresponds to setting $m = 0$). In this case equation (5.54) for $P(\vartheta)$ reduces to

$$l(l+1)\,\sin^2\vartheta\,P + \sin\vartheta\,\frac{\mathrm{d}}{\mathrm{d}\vartheta}\left(\sin\vartheta\,\frac{\mathrm{d}P}{\mathrm{d}\vartheta}\right) = 0\,, \tag{5.58}$$

where l is a positive integer or zero. Let us introduce a *new* independent variable x by

$$x = \cos\vartheta\,. \tag{5.59}$$

Then for any function $f(\vartheta)$ we have

$$\frac{\mathrm{d}f}{\mathrm{d}\vartheta} = \frac{\mathrm{d}f}{\mathrm{d}x}\,\frac{\mathrm{d}x}{\mathrm{d}\vartheta} = -\,\sin\vartheta\,\frac{\mathrm{d}f}{\mathrm{d}x}\,. \tag{5.60}$$

Substituting in (5.58), and noting that $\sin^2\vartheta = 1 - x^2$, we now get

$$l(l+1)(1-x^2)\,P + (1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x^2)\frac{\mathrm{d}P}{\mathrm{d}x}\right) = 0\,, \tag{5.61}$$

or

$$(1-x^2)\frac{\mathrm{d}^2P}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}P}{\mathrm{d}x} + l(l+1)\,P = 0\,. \tag{5.62}$$

This linear ordinary differential equation of second order is known as **Legendre's equation** (named after the French mathematician Adrien-Marie Legendre, 1752–1833). It has polynomial solutions of order l for *integral* values of l, and the solution is convergent for all x in the interval $-1 \leq x \leq 1$. (If l is *not* integral, the solutions do *not* converge for $x = \pm 1$.) The first few of these **Legendre polynomials of order** l, denoted $P_l(x)$, are ($l = 0, 1, 2, 3$)

$$P_0(x) = 1\,, \qquad P_1(x) = x\,, \qquad P_2(x) = \frac{3x^2-1}{2}\,, \qquad P_3(x) = \frac{5x^3-3x}{2}\,. \tag{5.63}$$

The following two remarks should be made: (i) the $P_l(x)$ alternately constitute **even** and **odd** polynomials in x, i.e., they satisfy

$$P_{l=2n}(-x) = P_{l=2n}(x) \qquad P_{l=2n+1}(-x) = -\,P_{l=2n+1}(x) \qquad n = 0, 1, 2, \dots\,,$$

and (ii) we have at the **boundaries** of the interval $-1 \leq x \leq 1$

$$P_l(x=1) = 1 \qquad\qquad P_l(x=-1) = (-1)^l\,.$$

When combining the **Legendre polynomials** with solution (5.56) for (5.52), we extract the general **axisymmetric solution** of **Laplace's equation**:

$$\phi(r,\vartheta) = \sum_{l=0}^{\infty} \left[\frac{a_l}{r^{(l+1)}} + b_l\, r^l \right] P_l(\cos\vartheta)\ , \tag{5.64}$$

where a_l and b_l are constants chosen to match the **boundary conditions** in any particular problem. Remember that the **uniqueness theorem** ensures that any set of constants that satisfy the boundary conditions provides a unique solution of **Laplace's equation**.

If we start to expand the general **axisymmetric solution** in powers of l and $\cos\vartheta$, we can ascribe a physical significance to the first few terms:

$$\phi(r,\vartheta) = \underbrace{b_0}_{\text{constant}} + \underbrace{\frac{a_0}{r}}_{\text{source/sink}} + \underbrace{b_1 r\cos\vartheta}_{\text{uniform stream}} + \underbrace{\frac{a_1\cos\vartheta}{r^2}}_{\text{dipole at origin}} + \ldots\ . \tag{5.65}$$

Example: rigid sphere in uniform stream

Consider a **rigid sphere** of radius a in a **uniform flow** of speed $U =$ constant. We arrange the **spherical polar coordinates** such that the ($\vartheta = 0$)–axis is aligned with the flow at infinity. Therefore the **flow velocity** at infinity is given by

$$\boldsymbol{u}_\infty = (U\cos\vartheta, -U\sin\vartheta, 0)^T\ , \tag{5.66}$$

and the corresponding **velocity scalar potential** is given by $\phi_\infty = Ur\cos\vartheta$ (this will be the first term we take from the general solution). The other **boundary condition** is that the radial component of the **flow velocity** (i.e., the normal derivative of ϕ) is zero at the surface of the sphere, i.e.,

$$u_r = \frac{\partial\phi}{\partial r} = 0 \quad \text{at} \quad r = a\ . \tag{5.67}$$

This boundary condition must be true for *all* ϑ, and this suggests that we need another term from the general solution, one which also has a $(\cos\vartheta)$–dependence that is linear. We try adding the **dipole term**:

$$\phi = Ur\cos\vartheta + \frac{A}{r^2}\cos\vartheta\ , \tag{5.68}$$

so

$$\frac{\partial\phi}{\partial r} = U\cos\vartheta - \frac{2A}{r^3}\cos\vartheta\ . \tag{5.69}$$

To satisfy the boundary condition (5.67) we need to have

$$A = \frac{1}{2}Ua^3\ , \tag{5.70}$$

so that the **velocity scalar potential** finally becomes

$$\phi = Ur\cos\vartheta\left(1 + \frac{a^3}{2r^3}\right)\ . \tag{5.71}$$

This solution satisfies **Laplace's equation** and the given **boundary conditions** (5.66) and (5.67). The resultant **flow velocity** $\boldsymbol{u} = \boldsymbol{\nabla}\phi$, i.e.,

$$\boldsymbol{u} = U\cos\vartheta\left(1 - \frac{a^3}{r^3}\right)\hat{\boldsymbol{e}}_r - U\sin\vartheta\left(1 + \frac{a^3}{2r^3}\right)\hat{\boldsymbol{e}}_\vartheta\ , \qquad (5.72)$$

is therefore the *unique solution*.

5.4.3 Cylindrical harmonics

Solutions of **Laplace's equation** in 2-D plane polar geometry, i.e., where

$$\frac{\partial\phi}{\partial z} = 0\ ,$$

are called **cylindrical harmonics**. The procedure for finding the **cylindrical harmonics** is similar to that for the **spherical harmonics**. One assumes a **separation of variables Ansatz** of the form

$$\phi(r,\varphi) = R(r)\,Q(\varphi)\ , \qquad (5.73)$$

substitutes into **Laplace's equation** expressed in **cylindrical polar coordinates** [see (2.15)],

$$(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial z^2} = 0\ , \qquad (5.74)$$

and divides by ϕ/r^2. Then one separates terms dependent on r and φ according to

$$\frac{r}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R}{\mathrm{d}r}\right) = -\frac{1}{Q}\frac{\mathrm{d}^2Q}{\mathrm{d}\varphi^2} = m^2\ , \qquad (5.75)$$

where we have chosen m^2 as the constant that arises in the separation procedure.

Although this appears a simple pair of linear ordinary differential equations of second order, there are in fact various types of solutions, depending on the value of m. For $m > 0$, the solution to the Q–equation is

$$Q(\varphi) = C\,\cos m\varphi + D\,\sin m\varphi\ , \qquad (5.76)$$

where C and D are arbitrary integration constants. If the problem is such that φ covers the entire range from 0 to 2π, then $Q(\varphi)$ must be single-valued, which implies that m must be *integral*. Without loss of generality, m can be taken as a positive integer or zero in this case. However, this condition, although generally satisfied, is *not always* satisfied. Nevertheless, in this course we will not be investigating the case where m is not integral.

In the case $m > 0$, the R–equation has the solution

$$R(r) = A\,r^m + B\,r^{-m}\ , \qquad (5.77)$$

with A and B arbitrary integration constants.

The case $m = 0$ has to be treated separately. Here the resultant expression is

$$\begin{aligned} \phi_0(r,\varphi) &= (A\,\varphi + B)\,(C\,\ln r + D) \\ &= \underbrace{E\,\varphi}_{\text{circulation around origin}} + \underbrace{F\,\ln r}_{\text{line source/sink}} \\ &\quad + G\,\varphi\,\ln r + H\ . \end{aligned} \tag{5.78}$$

Therefore, the final expression for the **velocity scalar potential** as expanded in terms of **cylindrical harmonics** is

$$\begin{aligned} \phi(r,\varphi) &= \phi_0(r,\varphi) + \sum_{m=1}^{\infty} r^m\,[\,a_m\,\cos m\varphi + b_m\,\sin m\varphi\,] \\ &\quad + \sum_{m=1}^{\infty} r^{-m}\,[\,c_m\,\cos m\varphi + d_m\,\sin m\varphi\,]\ . \end{aligned} \tag{5.79}$$

As before, *all* constants (including a_m, b_m, c_m and d_m) need to be adjusted so as to satisfy any given **boundary conditions**.

Example: rigid cylinder in uniform stream

Consider a **steady flow**, **uniform** and in the positive x–direction upstream, that runs past a **rigid cylinder** of radius a which is centred on the origin and "infinitely extended" in the z–direction.

At infinity (in both the negative and positive x–direction) we have $\boldsymbol{u} = U\,\boldsymbol{e}_x$, with $U =$ constant. In **cylindrical polar coordinates** this is expressed by

$$\boldsymbol{u}_\infty = (U\cos\varphi, -U\sin\varphi, 0)^T\ . \tag{5.80}$$

The corresponding **velocity scalar potential** is $\phi_\infty = U r\cos\varphi$. As in previous examples, on the surface of the cylinder we have the **boundary condition** that the **normal component** of the **flow velocity** needs to be zero (i.e., the normal derivative of ϕ), i.e.,

$$u_r = \frac{\partial\phi}{\partial r} = 0 \quad \text{at} \quad r = a\ , \tag{5.81}$$

and this must be true for *all* φ. Accordingly we choose a term from the general solution (5.79) which has a $(\cos\varphi)$–dependence that can dominate at *small* r:

$$\frac{c_1}{r}\cos\varphi\ . \tag{5.82}$$

This might seem strange, given the term $d_m\, r^{-m}\ \sin m\varphi$ in (5.79), but we have $\cos\varphi = \sin(\varphi - \pi/2)$ which can be expressed as a constant times $\sin\varphi$ using the double angle formula. In the present example, we also have to take into account that the region of flow around the cylinder is *not* simply connected! For any closed integration path enclosing the cylinder there will thus be a *non-zero* **circulation**.

Hence, we try the following **Ansatz**:

$$\phi = Ur\cos\varphi + \frac{c_1}{r}\cos\varphi + E\,\varphi\ . \tag{5.83}$$

The boundary condition (5.81) at the surface of the cylinder demands

$$u_r = \frac{\partial\phi}{\partial r} = U\cos\varphi - \frac{c_1}{r^2}\cos\varphi = 0 \quad \text{at} \quad r = a\ . \tag{5.84}$$

Therefore,

$$c_1 = Ua^2\ . \tag{5.85}$$

Also, the line integral along a circle C of constant radius L enclosing the cylinder, with line element $\mathrm{d}\boldsymbol{l} = L\mathrm{d}\varphi\,\hat{\boldsymbol{e}}_\varphi$, yields for the **circulation**

$$\begin{aligned}\Gamma &= \oint_C \boldsymbol{u}\cdot\mathrm{d}\boldsymbol{l} \ = \ L\int_0^{2\pi} u_\varphi\,\mathrm{d}\varphi \\ &= -L\int_0^{2\pi}\left[U\sin\varphi\left(1+\frac{a^2}{L^2}\right) - \frac{E}{L}\right]\mathrm{d}\varphi \\ &= 2\pi E\ ,\end{aligned} \tag{5.86}$$

so

$$E = \frac{\Gamma}{2\pi}\ . \tag{5.87}$$

Thus, the final solution for the **velocity scalar potential** satisfying the **boundary conditions** (5.80) and (5.81) is

$$\phi = Ur\cos\varphi\left(1+\frac{a^2}{r^2}\right) + \frac{\Gamma}{2\pi}\,\varphi\ . \tag{5.88}$$

The second term here corresponds to a line doublet term. It follows that the unique flow pattern has **flow velocity** given by $\boldsymbol{u} = \boldsymbol{\nabla}\phi$, i.e.,

$$\boldsymbol{u} = U\cos\varphi\left(1-\frac{a^2}{r^2}\right)\hat{\boldsymbol{e}}_r - U\sin\varphi\left(1+\frac{a^2}{r^2}\right)\hat{\boldsymbol{e}}_\varphi + \frac{\Gamma}{2\pi r}\,\hat{\boldsymbol{e}}_\varphi\ . \tag{5.89}$$

Chapter 6
Fluid equations of motion

6.1 Forces in fluids

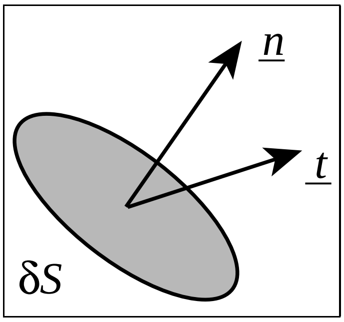
We now turn back to the study of fluid motions as determined by some **equations of motion**. We mean by this in particular (in the traditional Newtonian sense) an equation which relates the **acceleration** of the motion of a fluid to the **forces** that are generating the motion. We can include **viscous forces**, and thus we will be discussing fluids which are more realistic than the **ideal fluids** introduced earlier. The equations of motion which we will be deriving are called the (general) **Navier–Stokes equations**. These are usually coupled with the **continuity equation** that we derived earlier in chapter 3. Crucial to the derivation of the **Navier–Stokes equations** is the idea of applying the physical principle of **conservation of linear momentum** to any particular **fluid element**. In this scheme a fluid element can be thought analogous to a particle (or tennis ball), for which one can easily write down the equations of motion.

When we consider the forces acting on a fluid element, or blob of fluid, there are *two* very different kinds of forces. The first are called **body forces** and act throughout the whole volume of a fluid element. Gravitational forces and inertial forces such as centrifugal or Coriolis forces (the latter named after the French mathematician Gaspard Gustave de Coriolis, 1792–1843) are examples of body forces, as are electromagnetic forces (if the fluid can carry an electric charge as, e.g., liquid mercury). The second kind of forces exist by virtue of the fluid element actually being surrounded by other fluid elements and are called **surface forces**. For a fluid element in the body of a fluid, the rest of the fluid can exert forces *only* by contact; in other words, only at the surface of the fluid element. Fluid elements adjoining a boundary will also experience surface forces exerted by the boundary. The *crucial point* about surface forces, which may not be self-evident, is that they can have a component which is **tangential** to the surface, apart from the component which is **normal**. Of course, in our definition of an **ideal fluid** we explicitly excluded the possibility of **tangential stresses**. As we will subsequently see, **viscous forces** are associated in particular with **tangential stresses**, though, in general, they also have a component corresponding to a special kind of **normal stresses**.

Note that most of what we discuss is actually applicable not just to **fluids** but to any **deformable continuous medium** such as, e.g., rubber, plastic or metal.

6.1.1 Stress vector

Consider in some domain $D \subset \mathbb{R}^3$ of a deformable continuous medium a small geometrical **surface element**, of area δS, which has **unit normal** $\boldsymbol{n}$ at some position $\boldsymbol{r}$.



The **force** exerted *on* this surface element by the medium into which $\boldsymbol{n}$ is pointing is assumed to be of the form

$$\boldsymbol{t}\,\delta S\ , \tag{6.1}$$

which thus defines the **stress vector**, $\boldsymbol{t}$. We want to *assume* that the **stress vector** can be expressed as a *linear* function of the **normal vector**. Then, by definition, we have in the case of **inviscid fluids** that $\boldsymbol{t} = -\,p(t,\boldsymbol{r})\,\boldsymbol{n}$, i.e., the **stress vector** acts along the **normal direction** only. On the other hand, we have already argued above that in the case of **viscous fluids** we expect the **stress vector** to have *both* **tangential** as well as **normal components**.

6.1.2 Stress tensor

For any **deformable continuous medium** the **stress tensor** encodes the transfer rate of **linear momentum** across contact surfaces between neighbouring **volume elements** which is due to **molecular motions** within the medium. It is standard to denote the **stress tensor** by $\boldsymbol{T}$. Using index notation, at any point of a given domain $D \subset \mathbb{R}^3$ in the medium its nine components T_{ij} are defined as follows:

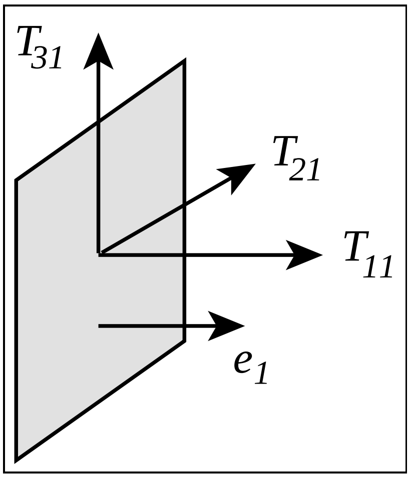
> T_{ij} *is the* i*-th component of stress on a surface element* δS *which has a normal* $\boldsymbol{n}$ *pointing in the* j*-direction.*

For example, in a **Cartesian coordinate system** T_{xz} (i.e., T_{13}) is the stress acting in the x–direction on a surface element whose normal is pointing in the z–direction.

The trace of the **stress tensor** yields, up to a constant of proportionality, what one defines as the **mechanical pressure**, or **mean normal stress**, within a deformable continuous medium. More precisely, we have

$$P := -\,\frac{1}{3}\,\mathrm{Tr}\,\boldsymbol{T} = -\,\frac{1}{3}\,(T_{11} + T_{22} + T_{33})\ . \tag{6.2}$$

Clearly, the physical dimension of $\boldsymbol{T}$ is $[\,\text{pressure}\,]$, i.e., $[\,\text{force}\,]\,[\,\text{area}\,]^{-1}$, which in the MKS-system is be expressed by $[\,\text{mass}\,]\,[\,\text{length}\,]^{-1}\,[\,\text{time}\,]^{-2}$. Hence, the SI unit for $\boldsymbol{T}$ is 1 kg m^{-1} s^{-2}, or 1 Pascal (Pa).



As understanding the **stress tensor** is the *key* to formulating the equations of motion in **fluid dynamics**, we want to take a look at a couple of its properties.

The definition just introduced of the **stress tensor** gives the forces across a surface when that surface has a **normal** in one of the coordinate directions. But obviously we would like to be able to treat more general cases. In particular, we would like to have an expression for the **stress vector** $\boldsymbol{t}$ acting on some **surface element** with a **normal** $\boldsymbol{n}$ that is aligned with an *arbitrary* spatial direction. Above we assumed that the general relation between $\boldsymbol{t}$ and $\boldsymbol{n}$ should be *linear*. The required expression is then the following:

For a surface element δS in a fluid, with unit normal $\boldsymbol{n}$, the stress vector $\boldsymbol{t}$ has components t_i given by (employing the summation convention)

$$t_i = T_{ij}\,n_j \; . \tag{6.3}$$

Relative to a **Cartesian coordinate basis** the proof of this is as follows:[1] Consider a surface element δS to be the large face of a **tetrahedron** within a deformable continuous medium, whose other faces are parallel to the x–y, y–z, and z–x coordinate planes, respectively. This tetrahedron shall represent a material volume element.

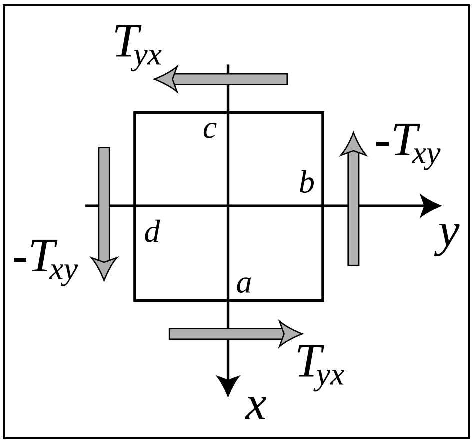
[1] If the proof holds with respect to one coordinate basis, it will hold with respect to all coordinate bases.

We now wish to apply the principle of **conservation of linear momentum** to the medium within the tetrahedral volume element at some instant of time. We collect together the i-th component of the **forces** *on* the tetrahedron. By definition the force exerted by the surrounding medium on the main face is $t_i\,\delta S$. The i-th component of **stress** exerted by the surrounding fluid on the face which is parallel to the y–z plane is $-T_{ix}$, because that face has unit normal $-\boldsymbol{e}_x$. The area of this face is given by $n_x\,\delta S$ (from thinking about the area of the projection of δS onto the y–z plane), so the final i-th component of **stress** on the face parallel to the y–z plane is $-T_{ix}\,n_x\,\delta S$. Analogously we find $-T_{iy}\,n_y\,\delta S$ and $-T_{iz}\,n_z\,\delta S$ for the faces parallel to the z–x and x–y planes, respectively. Thus the **total force** (i-th component) exerted on the medium enclosed by the tetrahedron is

$$\underbrace{(t_i - T_{ij}\,n_j)\,\delta S}_{\text{net surface force}} \quad + \quad \underbrace{\rho\,g_i\,\delta V}_{\text{net body force}} \quad , \tag{6.4}$$

where we have added a gravitational **body force**, $\rho\,\boldsymbol{g}\,\delta V$, with δV the volume of the tetrahedron. Now the **acceleration** felt by the medium in the volume element equals the force acting on it divided by its mass (which is $\rho\,\delta V$). Making a **dimensional analysis**, we can take the linear dimension of the tetrahedron to be length L, and so its **volume** δV scales as L^3 while the **area** δS scales as L^2. Hence, we see that the **acceleration** due to the net surface forces scales as L^{-1}. Clearly, in the limit $L \to 0$ the acceleration must remain *finite*, and that is only possible if the net surface force is *zero*. This implies that the **stress vector** is given by $t_i = T_{ij}\,n_j$, as claimed.

Using a similar argument we can show that the **stress tensor** has to be **symmetric**. Consider in a **Cartesian coordinate basis** a **unit cube**, and the **tangential forces** it feels in the x–y plane. We label the faces with normals parallel to this plane by a, b, c and d.



To understand the forces, as shown in the figure, consider face a. The force, as drawn, is in the y–direction, and the face normal is in the x–direction, so that the tangential force is T_{yx} (remember we are considering a unit cube). We can then find the **total couple** of these forces about the origin as $T_{xy} - T_{yx}$. The couple is the force times the distance from the point about which we wish to consider a rotation. Again, we consider what happens as the cube gets smaller. Let the linear dimension of any of its sides be length L. The **couple** scales as L^3: one power of L from **lever arm length**, two from force being proportional to **area** of a side. The **moment of inertia** of the cube scales as L^5: three powers of L from mass being proportional to **volume** and two from **radius of gyration**. Applying the principle of **conservation of angular momentum** — the rate of change in time of angular momentum (angular frequency times moment of inertia) is equal to the applied couple — we can conclude that the **angular frequency** scales as L^{-2}, and in order for this to remain *finite* in the limit $L \to 0$ we must have a total couple of *zero*, i.e., $T_{xy} = T_{yx}$. Hence, we find:

> *The stress tensor* $\boldsymbol{T}$ *is symmetric, i.e., in index form*
>
> $$T_{ij} = T_{ji} \, . \tag{6.5}$$

6.2 Fluid equations of motion

6.2.1 Cauchy's equations of motion

We can now move on to establish the general **equations of motion** for any **deformable continuous medium**, which was first obtained by the French mathematician Augustin Louis Cauchy (1789–1857). In order to do so, we will employ again a formulation that makes use of the concept of **balance equations** which we introduced in chapter 3 (cf. subsection 3.1.2). In particular, our aim is to write down a balance equation for **linear momentum** and thus describe how linear momentum is transferred between neighbouring volume elements of any deformable medium for which the **continuum hypothesis** holds (cf. chapter 1, subsection 1.4.2).

Let us thus consider within a deformable continuous medium a fixed region G that is bounded by a closed surface ∂G. As **linear momentum** is a *vector-valued* extensive quantity, the formulation of a balance equation for this quantity requires the introduction of a vector-valued **linear momentum density**. In the present case this is just the product between the mass density and the flow velocity of the deformable continuous medium in question, i.e.,

$$\rho \boldsymbol{u} \ ,$$

and so its physical dimension is $[\,\text{mass}\,]\,[\,\text{length}\,]^{-2}\,[\,\text{time}\,]^{-1}$. In index notation the linear momentum density is expressed by

$$\rho u_i \ .$$

(Note that, hence, linear momentum density is equivalent to mass current density.)

Next, we need to introduce a tensor-valued **linear momentum current density**, denoted by $\boldsymbol{\Pi}$, and of physical dimension $[\,\text{mass}\,]\,[\,\text{length}\,]^{-1}\,[\,\text{time}\,]^{-2}$. This tensor is constructed from two parts: a **convective part** simply due to the mechanical transport of linear momentum by different volume elements as they move from place to place, and a molecular **conductive part** due to stresses acting across the contact surfaces between neighbouring volume elements. The linear momentum current density can thus be written as[2]

$$\boldsymbol{\Pi} = (\rho \boldsymbol{u}) \otimes \boldsymbol{u} - \boldsymbol{T} \ ,$$

with $\boldsymbol{T}$ the **stress tensor** of the deformable continuous medium and the mathematical symbol "$\otimes$" denoting the **tensor product** between vector fields in $\mathbb{R}^3$. In index notation this is just

$$\Pi_{ij} = \rho u_i u_j - T_{ij} \ .$$

Finally, we assume that the deformable continuous medium is exposed to **body forces** (e.g., gravitational, inertial, or electromagnetic), which thus contribute a **linear momentum generation rate density** that we conveniently write as

$$\rho \boldsymbol{f}_{\text{body}} \ ;$$

its physical dimension is $[\,\text{mass}\,]\,[\,\text{length}\,]^{-2}\,[\,\text{time}\,]^{-2}$. Note that the physical dimension of $\boldsymbol{f}_{\text{body}}$ itself is $[\,\text{acceleration}\,]$, i.e., $[\,\text{length}\,]\,[\,\text{time}\,]^{-2}$. In index notation the linear momentum generation rate density due to body forces is written as

$$\rho f_{\text{body},i} \ .$$

We are now in a position to write down a vector-valued analogue of the balance equation (3.17); applied to **linear momentum**. For a *fixed* region G within a deformable continuous medium, bounded by a closed surface ∂G, the rate of

[2]Note that the sign of $\boldsymbol{T}$ is such that when for a surface element with normal $\boldsymbol{n}$ we have $T_{ij} n_i n_j < 0$ (employing the summation convention), then there is a net push on the surface element. Otherwise, when $T_{ij} n_i n_j > 0$, there is a net pull.

change in time of linear momentum contained in G, due to linear momentum inflow across ∂G and body forces acting throughout G, is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_G \rho\boldsymbol{u}\,\mathrm{d}V = -\iint_{\partial G}\boldsymbol{\Pi}\cdot\boldsymbol{n}\,\mathrm{d}A + \iiint_G \rho\boldsymbol{f}_{\mathrm{body}}\,\mathrm{d}V\ . \qquad (6.6)$$

In index notation this balance equation reads (remember the summation convention, here and below!)

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_G \rho u_i\,\mathrm{d}V = -\iint_{\partial G}\Pi_{ij}\,n_j\,\mathrm{d}A + \iiint_G \rho f_{\mathrm{body},i}\,\mathrm{d}V\ . \qquad (6.7)$$

Now, in line with the procedure described in subsection 3.1.2, we can move the time derivative on the left-hand side of (6.6) under the volume integral (where it becomes a partial derivative), and also convert the flux integral on the right-hand side via a *generalised version of* **Gauß' integral theorem** (2.32) into a volume integral. Thus, as (6.6) has to hold for any arbitrary fixed G (bounded by ∂G), we find that the differential form of the **balance equation for linear momentum** in a deformable continuous medium is given by

$$\frac{\partial(\rho\boldsymbol{u})}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\Pi} = \rho\boldsymbol{f}_{\mathrm{body}}\ . \qquad (6.8)$$

In index notation, with respect to a **Cartesian coordinate basis**, this reads

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial\Pi_{ij}}{\partial x_j} = \rho f_{\mathrm{body},i}\ . \qquad (6.9)$$

Finally, employing that $\boldsymbol{\Pi} = (\rho\boldsymbol{u})\otimes\boldsymbol{u} - \boldsymbol{T}$, and using the **continuity equation** for mass in the form (3.22), we obtain from (6.8) **Cauchy's equations of motion**:

$$\rho\,\frac{\partial\boldsymbol{u}}{\partial t} + \rho\,(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} - \boldsymbol{\nabla}\cdot\boldsymbol{T} = \rho\boldsymbol{f}_{\mathrm{body}}\ . \qquad (6.10)$$

In index notation, with respect to a **Cartesian coordinate basis**, this is

$$\rho\,\frac{\partial u_i}{\partial t} + \rho\,u_j\,\frac{\partial u_i}{\partial x_j} - \frac{\partial T_{ij}}{\partial x_j} = \rho f_{\mathrm{body},i}\ . \qquad (6.11)$$

Up to this stage we have been working in the context of *any* **deformable continuous medium**. The next stage is to specify the *particular form* of the **stress tensor** which is appropriate for a **fluid**.

6.2.2 Stress tensor for Newtonian viscous fluids

In this course we want to restrict ourselves to **Newtonian viscous fluids**. As briefly referred to in chapters 1 and 3, these are fluids for which it is a good approximation to *assume* that **viscous stresses** within the fluid are *proportional to the velocity gradient*. On physical grounds it is clear that viscous stresses should occur neither in a fluid at rest nor in a fluid in rigid rotation. Hence, we can *define* an intrinsically isotropic fluid (one with no preferred direction *a priori*) to be a **Newtonian viscous fluid** if its **stress tensor** has the form

$$\boldsymbol{T} = -\,(p - \zeta\theta)\,\mathbf{g} + 2\mu\,\boldsymbol{\sigma}\ . \qquad (6.12)$$

Here p denotes the **thermodynamical pressure**, $\theta = \boldsymbol{\nabla}\cdot\boldsymbol{u}$ the **expansion/-contraction scalar**, $\boldsymbol{\sigma} = (\boldsymbol{\nabla}\otimes\boldsymbol{u})_{\rm sym} - \frac{1}{3}\,\mathbf{g}\,(\boldsymbol{\nabla}\cdot\boldsymbol{u})$ the **shear tensor**, and $\mathbf{g}$ the **metric tensor** in $\mathbb{R}^3$.[3] Moreover, this expression introduces $\zeta \geq 0$ as the **coefficient of volume viscosity** and $\mu \geq 0$ as the **coefficient of shear viscosity**. In general, both these scalar-valued quantities are thermodynamical functions, say of the **specific entropy** s and the **specific volume** $1/\rho$. Their physical dimension is $[\,\text{mass}\,]\,[\,\text{length}\,]^{-1}\,[\,\text{time}\,]^{-1}$.

In index notation, with respect to a **Cartesian coordinate basis**, the components of the **stress tensor** for a **Newtonian viscous fluid** are given by

$$T_{ij} = -\left(p - \zeta\,\frac{\partial u_k}{\partial x_k}\right)\delta_{ij} + \mu\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} - \frac{2}{3}\,\frac{\partial u_k}{\partial x_k}\,\delta_{ij}\right)\ . \qquad (6.13)$$

We note that this expression is symmetric, $T_{ij} = T_{ji}$, as required from the discussion above. Hence, (6.13) represents only six independent components rather then nine.

From (6.12), together with (6.2), we see that the **mechanical pressure** at a given point within a **Newtonian viscous fluid** is given by

$$P = p - \zeta\theta\ .$$

We can discuss further the form of $\boldsymbol{T}$ that we have chosen. The pressure term corresponds to an isotropic (same in all directions) compression. The second part is proportional to the shear tensor; in other words, it is proportional to the volume preserving change of shape of a fluid element. Note that the viscous terms *disappear* if there is *no* isotropic expansion/contraction of any fluid elements nor any anisotropic deformation. In particular, in the hydrostatic limit ($\boldsymbol{u} = \boldsymbol{0}$) (6.12) reduces to yield the **hydrostatic stress tensor**

$$\boldsymbol{T} = -\,p\,\mathbf{g}\ , \qquad (6.14)$$

or, in index notation with respect to a **Cartesian coordinate basis**,

$$T_{ij} = -\,p\,\delta_{ij}\ . \qquad (6.15)$$

The forms (6.14) and (6.15) also apply for **inviscid fluids**.

We can finally note that *not* all viscous fluids are of the Newtonian kind. **Non-Newtonian viscous fluids** can have very strange properties, such as a shear viscosity which decreases with the velocity shear (e.g., non-drip paint). Other examples of non-Newtonian viscous fluids are blood, and fluids with higher molecular weights (e.g., polymers such as molten plastics).

[3]Remember that in a coordinate basis of $\mathbb{R}^3$ the metric tensor $\mathbf{g}$ has components defined by $g_{ij} := \boldsymbol{e}_i\cdot\boldsymbol{e}_j$, $i, j = 1, 2, 3$. Thus, in a Cartesian coordinate basis we have $g_{ij} = \delta_{ij}$.

6.3 Navier–Stokes equations

6.3.1 Navier–Stokes equations for compressible viscous fluids

To obtain the equations of motion for a general **Newtonian viscous fluid**, we substitute the stress tensor (6.12) into **Cauchy's equations of motion** (6.10). Thus we find the general **Navier–Stokes equations**

$$\rho\,\frac{\partial \boldsymbol{u}}{\partial t} + \rho\,(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} + \boldsymbol{\nabla}(p-\zeta\theta) - 2\boldsymbol{\nabla}\cdot(\mu\,\boldsymbol{\sigma}) = \rho \boldsymbol{f}_{\rm body}\ , \tag{6.16}$$

where $\zeta \geq 0$ and $\mu \geq 0$ have to be given as thermodynamical functions.

However, in many applications concerning **Newtonian viscous fluid** one can treat these coefficients as *constant parameters*. In this case, upon combining (6.16) with the **continuity equation** (3.22), the equations of motion of **fluid dynamics** are given by

Navier–Stokes equations for compressible Newtonian viscous fluids:

$$\rho\,\frac{\partial \boldsymbol{u}}{\partial t} + \rho\,(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}$$
$$-\,(\zeta+\frac{1}{3}\,\mu)\,\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{u}) - \mu\,(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\boldsymbol{u} + \boldsymbol{\nabla}p = \rho\boldsymbol{f}_{\rm body} \tag{6.17}$$
$$\frac{\partial \rho}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\rho + \rho\,(\boldsymbol{\nabla}\cdot\boldsymbol{u}) = 0\ . \tag{6.18}$$

In index notation, with respect to a **Cartesian coordinate basis**, these are

$$\rho\,\frac{\partial u_i}{\partial t} + \rho\,u_j\,\frac{\partial u_i}{\partial x_j}$$
$$-\,(\zeta+\frac{1}{3}\,\mu)\,\frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right) - \mu\,\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{\partial p}{\partial x_i} = \rho f_{{\rm body},i} \tag{6.19}$$
$$\frac{\partial \rho}{\partial t} + u_i\,\frac{\partial \rho}{\partial x_i} + \rho\,\frac{\partial u_i}{\partial x_i} = 0\ . \tag{6.20}$$

We have thus derived a coupled system of four non-linear partial differential equations of *second order* for (i) the three components of $\boldsymbol{u}$ and (ii) ρ. To close this system of equations we need to prescribe both p and $\boldsymbol{f}_{\rm body}$ as functions of the independent variables t and $\boldsymbol{r}$. **Unique solutions** can then be obtained upon specification of the **initial conditions** and **boundary conditions** which $\boldsymbol{u}$ and ρ shall satisfy.

6.3.2 Navier–Stokes equations for incompressible viscous fluids

In all the examples of Newtonian viscous fluids we discuss in this course we restrict ourselves to the incompressible case, i.e., $\theta = \boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$. Thus, from (6.12), the stress tensor for an **incompressible Newtonian viscous fluid** is given by

$$\boldsymbol{T} := -\,p\,\mathbf{g} + 2\mu\,\boldsymbol{\sigma}\ , \tag{6.21}$$

or, in index notation, with respect to a **Cartesian coordinate basis**,

$$T_{ij} = -\,p\,\delta_{ij} + \mu\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)\ . \tag{6.22}$$

Moreover, the only **body force** we want to consider is a **gravitational force**, i.e.,

$$\boldsymbol{f}_{\text{body}} = \boldsymbol{g}\ .$$

Then the equations of motion of **fluid dynamics** become the

Navier–Stokes equations for incompressible Newtonian viscous fluids:

$$\rho\,\frac{\partial\boldsymbol{u}}{\partial t} + \rho\,(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} - \mu\,(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\boldsymbol{u} = -\,\boldsymbol{\nabla}p + \rho\,\boldsymbol{g} \tag{6.23}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0\ , \tag{6.24}$$

which Navier first derived in 1822, and Stokes obtained independently in 1845. In index notation, with respect to a **Cartesian coordinate basis**, these are

$$\rho\,\frac{\partial u_i}{\partial t} + \rho\,u_j\,\frac{\partial u_i}{\partial x_j} - \mu\,\frac{\partial^2 u_i}{\partial x_j\partial x_j} = -\,\frac{\partial p}{\partial x_i} + \rho\,g_i \tag{6.25}$$

$$\frac{\partial u_i}{\partial x_i} = 0\ . \tag{6.26}$$

We are thus dealing with a coupled system of four non-linear partial differential equations of *second order* for the three components of $\boldsymbol{u}$. As regards $\boldsymbol{u}$ this system is therefore *overdetermined*. Nevertheless, to solve this system we need to give both p and $\boldsymbol{g}$ as functions of the independent variables t and $\boldsymbol{r}$. **Unique solutions** can be obtained upon specification of the **initial conditions** and **boundary conditions** that $\boldsymbol{u}$ shall satisfy.

From chapter 2 we know that with respect to a **Cartesian coordinate basis** (only!) the vector identity

$$(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\boldsymbol{u} = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{u}) - \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{u})$$

holds [cf. (2.31)]. Using this, we can rewrite the **Navier–Stokes equations** (6.23) as

$$\rho\,\frac{\partial\boldsymbol{u}}{\partial t} + \rho\,(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} + \mu\,\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{u}) = -\,\boldsymbol{\nabla}p + \rho\,\boldsymbol{g}\ . \tag{6.27}$$

Obviously, we can ask what happens when the fluid is **inviscid** (zero shear viscosity μ). Then

$$\rho\,\frac{\partial\boldsymbol{u}}{\partial t} + \rho\,(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} = -\,\boldsymbol{\nabla}p + \rho\,\boldsymbol{g} \tag{6.28}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0\ , \tag{6.29}$$

and it is no surprise to recognize **Euler's equations** (3.45) and (3.46) describing the motions of **incompressible inviscid fluids**, which we derived from first principles earlier on. Again, in index notation with respect to a **Cartesian coordinate basis** these are

$$\rho\,\frac{\partial u_i}{\partial t} + \rho\, u_j\,\frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \rho\, g_i \tag{6.30}$$

$$\frac{\partial u_i}{\partial x_i} = 0\,. \tag{6.31}$$

Here we have a coupled system of four non-linear partial differential equations of *first order* for the three components of $\boldsymbol{u}$, and so this system is also *overdetermined*. With given p and $\boldsymbol{g}$ as functions of t and $\boldsymbol{r}$ we can obtain **unique solutions** upon specification of the **initial conditions** and **boundary conditions** that $\boldsymbol{u}$ shall satisfy.

6.4 Kelvin's circulation theorem

We can use the **Navier–Stokes equations** for incompressible Newtonian viscous fluids to derive results about the role of **viscosity** in the evolution of **circulation** and **vorticity** in a fluid.

Let us first address the **circulation**. Consider a closed material curve made up of the *same* fluid elements for *all* time, so that the circuit $C(t)$ so defined changes in response to the motion of the fluid (it "moves with the fluid"). Then the rate of change of the circulation around this circuit, *as the curve moves with the fluid*, is

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} = \frac{\mathrm{D}}{\mathrm{D}t}\oint_C \boldsymbol{u}\cdot\mathrm{d}\boldsymbol{l}\,, \tag{6.32}$$

or

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} = \oint_C \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\cdot\mathrm{d}\boldsymbol{l} + \oint_C \boldsymbol{u}\cdot\frac{\mathrm{D}\mathrm{d}\boldsymbol{l}}{\mathrm{D}t}\,. \tag{6.33}$$

We will be using the **Navier–Stokes equations** to modify the first term, and for the second term we use a result from our earlier discussion on the analysis of fluid motion, namely that

$$\left(\frac{\mathrm{D}\mathrm{d}\boldsymbol{l}}{\mathrm{D}t}\right)_i = u_{ij}\mathrm{d}l_j\,, \tag{6.34}$$

where u_{ij} is the matrix of all the gradients of the components of the fluid velocity,

$$u_{ij} = \frac{\partial u_j}{\partial x_i}\,. \tag{6.35}$$

Then, using index notation,

$$\boldsymbol{u}\cdot\frac{\mathrm{D}\mathrm{d}\boldsymbol{l}}{\mathrm{D}t} = u_i\left(\frac{\mathrm{D}\mathrm{d}\boldsymbol{l}}{\mathrm{D}t}\right)_i \tag{6.36}$$

$$= u_i u_{ij}\mathrm{d}l_j \tag{6.37}$$

$$= u_i(\partial_j u_i)\mathrm{d}l_j \tag{6.38}$$

$$= \mathrm{d}l_j \partial_j (\tfrac{1}{2} u_i u_i) \tag{6.39}$$
$$= \mathrm{d}l_j \partial_j (\tfrac{1}{2} u^2) \tag{6.40}$$
$$= \mathrm{d}\boldsymbol{l} \cdot \boldsymbol{\nabla} (\tfrac{1}{2} u^2) \,, \tag{6.41}$$

since by the summation convention $u_i u_i = u_1^2 + u_2^2 + u_3^2 = u^2$.

Now, using the **Navier–Stokes equations** (with **kinematic viscosity** $\nu := \mu/\rho$),

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} = \oint_C \left[\boldsymbol{g} - \frac{1}{\rho} \boldsymbol{\nabla} p + \nu \, (\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}) \boldsymbol{u} + \boldsymbol{\nabla} (\tfrac{1}{2} \, u^2) \right] \cdot \mathrm{d}\boldsymbol{l} \,, \tag{6.42}$$

so

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} = \oint_C \left[\boldsymbol{\nabla} \left(- \Phi - \frac{p}{\rho} + \tfrac{1}{2} \, u^2 \right) + \nu \, (\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}) \boldsymbol{u} \right] \cdot \mathrm{d}\boldsymbol{l} \,, \tag{6.43}$$

using the fact that ρ is constant if the fluid is incompressible, and also assuming the existence of a **scalar potential** Φ for the **gravitational acceleration** $\boldsymbol{g}$.

Next, by Stokes' integral theorem (2.33), $\oint \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l} = 0$, if a vector-valued function $\boldsymbol{A}$ can be written as the gradient of a scalar-valued function ϕ. Consequently, in the equation above, the gradient terms integrate to zero, leaving

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} = \oint_C \nu \, (\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}) \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{l} \,. \tag{6.44}$$

Thus, we can finally state

Kelvin's circulation theorem:
The circulation around a closed material curve (one that moves with the fluid) in a inviscid ($\nu = 0$), incompressible fluid under conservative body forces (such as gravity) is constant.

This theorem does *not* require that the fluid domain $D \subset \mathbb{R}^3$ is simply connected, just that the closed circuit is wholly in the fluid.

The theorem has an important corollary: the **Cauchy–Lagrange theorem**. Consider an **inviscid, incompressible fluid** of mass density ρ = constant which moves in the presence of a **conservative body force**. If a portion of the fluid is *initially* in **irrotational motion**, then that portion will always *remain* in **irrotational motion**.

To prove this, suppose that at some later time the vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$ were *not* identically zero throughout the considered portion of fluid. Using Stokes' integral theorem (2.33), we have

$$\Gamma = \oint_C \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{l} = \int\!\!\int_S \boldsymbol{\omega} \cdot \boldsymbol{n} \, \mathrm{d}A \,, \tag{6.45}$$

and so it would be possible to choose some small closed material circuit around which the circulation would be non-zero. But this would violate **Kelvin's circulation theorem**, because the circulation around such a circuit must have originally been zero (using Stokes' integral theorem and the assumption of initially zero vorticity). It follows that our assumption (that a region of non-zero vorticity has developed in the portion of fluid in question) must be false, and the theorem is proved.

An almost trivial consequence of this result is that if any irrotational fluid motion is developed from rest, then the flow will remain irrotational (since a static fluid is trivially irrotational).

It is also clear from our result for $\mathrm{D}K/\mathrm{D}t$ that **vorticity** can be introduced into an otherwise irrotational flow if **viscous interactions** become dynamically important. This is especially of interest when an obstacle starts *moving* in a fluid, since this is the time when the boundary layer is created, and, by definition, the time when viscous effects are most important in determining the flow close to the obstacle.

6.5 Helmholtz's vortex theorems

Kelvin's circulation theorem also has implications for the evolution of vorticity in a fluid. But first we must define a couple of new terms.

A **vortex line** is, at any particular instant of time t, a curve which at all points has the same *direction* as the vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{u}$. If in **Cartesian coordinates** a vortex line is specified parametrically as $x = x(s)$, $y = y(s)$, and $z = z(s)$, then the three ordinary differential equations of first-order for the vortex line are

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \lambda\,\omega_x\,, \quad \frac{\mathrm{d}y}{\mathrm{d}s} = \lambda\,\omega_y\,, \quad \frac{\mathrm{d}z}{\mathrm{d}s} = \lambda\,\omega_z\,, \qquad (6.46)$$

with λ an arbitrary constant, or equivalently,

$$\omega_y\mathrm{d}z = \omega_z\mathrm{d}y\,, \quad \omega_z\mathrm{d}x = \omega_x\mathrm{d}z\,, \quad \omega_x\mathrm{d}y = \omega_y\mathrm{d}x\,, \qquad (6.47)$$

at any particular time t. (Note that the method for obtaining fluid vortex lines follows a procedure analogous to obtaining fluid **streamlines**.)

A **vortex tube** is bounded by the set of vortex lines which pass through some simple closed curve in space.

Now consider a fluid to which we can apply **Kelvin's circulation theorem**. Suppose that there is an *inviscid*, incompressible fluid of constant density moving subject to a conservative body force. Then:

Fluid elements that lie on a vortex line at some instant of time continue to lie on a vortex line, i.e., vortex lines "move with the fluid". It follows that vortex tubes also move with the fluid in a similar way to vortex lines.

There is a further result concerning vortex tubes:

Helmholtz' first vortex theorem:
The "strength" of a vortex tube Γ*, defined as*

$$\Gamma = \int\!\!\int_S \boldsymbol{\omega}\cdot\boldsymbol{n}\,\mathrm{d}A\,, \tag{6.48}$$

is the same for all cross-sections S *of a vortex tube. And the strength of a vortex tube is constant as it moves with the fluid.*

The proof of **Helmholtz' first vortex theorems** proceeds as follows. A **vortex surface** is a surface having $\boldsymbol{\omega}$ everywhere tangent. A vortex line is thus the *intersection* of two vortex surfaces. Consider the fluid elements which make up one of these surfaces, S, at a time $t = 0$. Consider a closed curve C made up of fluid elements which lie in the vortex surface S and span a surface $S*$ (a subset of S). Since $\boldsymbol{\omega}$ is tangential everywhere over $S*$, it follows, from Stokes' integral theorem, that at $t = 0$ the circulation around C is zero,

$$\Gamma = \int_C \boldsymbol{u}\cdot\mathrm{d}\boldsymbol{l} = \int\!\!\int_{S*} \boldsymbol{\omega}\cdot\boldsymbol{n}\,\mathrm{d}A = 0\,, \tag{6.49}$$

because all over $S*$ we have $\boldsymbol{\omega}\cdot\boldsymbol{n} = 0$. By **Kelvin's circulation theorem**, the circulation around C will remain zero even as C deforms when the fluid moves. This is true for all such curves in S, so that, using Stokes' integral theorem again, $\boldsymbol{\omega}\cdot\boldsymbol{n} = 0$ at all points in a vortex surface at all later times. In other words, a vortex surface at some time will remain a vortex surface as time goes on. The final step of the proof is to note that therefore the intersection of two vortex sheets will remain a vortex line.

Before proceeding to **Helmholtz' second vortex theorems**, we note that the fact that the strength of a vortex tube is independent of the cross-section used to calculate it is a consequence of the fact that the vorticity (being the curl of a vector) is divergence-free. Consider a volume consisting of a finite length of a vortex tube, with bounding cross-sections S_1 and S_2. We apply Gauß' integral theorem (2.32) to this volume, obtaining

$$\int\!\!\int_S \boldsymbol{\omega}\cdot\mathrm{d}\boldsymbol{A} = \int\!\!\int_{S_1} \boldsymbol{\omega}\cdot\mathrm{d}\boldsymbol{A} - \int\!\!\int_{S_2} \boldsymbol{\omega}\cdot\mathrm{d}\boldsymbol{A} = \int\!\!\int\!\!\int_V \boldsymbol{\nabla}\cdot\boldsymbol{\omega}\,\mathrm{d}V = 0\,, \tag{6.50}$$

where we use the fact that the *sides* of the vortex tube make *no* contribution since $\boldsymbol{\omega}$ is tangential there, and also we take account of the fact that one of the normals of the cross-sections is pointing *into* the volume, rather than out of it. It follows then that

$$\Gamma_{S_1} = \Gamma_{S_2}\,, \tag{6.51}$$

i.e., the strength of the flux tube is independent of the cross-section.

Now let us turn to the claim that the strength Γ is independent of time. Consider a circuit C of the flux tube comprising fluid elements which lie on the wall of the vortex tube and encircle it. By Stokes' integral theorem, Γ is just the circulation around C,

$$\Gamma = \int\!\!\int_S \boldsymbol{\omega}\cdot\boldsymbol{n}\,\mathrm{d}A = \oint_C \boldsymbol{u}\cdot\mathrm{d}\boldsymbol{l}\,, \tag{6.52}$$

which, by **Kelvin's circulation theorem**, is constant as time proceeds because the vortex tube, and hence the circuit C, moves with the fluid.

Consider a thin **vortex tube**, so that the strength is just $\boldsymbol{\omega}\,\delta S$ (vorticity is roughly constant across the vortex tube). But the fluid within the vortex tube must conserve its volume, so that, if the tube *lengthens*, it follows that δS *decreases* and consequently the vorticity $\boldsymbol{\omega}$ within the tube must *increase*: stretching the vortex tube by fluid motion intensifies the local vorticity. An example of this is a **tornado**, where strong thermal updraughts produce intense stretching of vortex tubes, and hence spectacularly destructive rotary motions. One can also observe the tipping over of the **funnel cloud** as the thunder clouds move on, which is an illustration of the first vortex theorem that vortex lines/tubes move with the fluid.

6.6 Bernoulli's streamline theorem (again!)

With the derivation of the **Navier–Stokes equations**, and its consequences for **circulation** and **vorticity**, behind us, we can go on to more examples of fluid flows — some elementary, some more complex.

As a reminder, we state again the **Navier–Stokes equations** for **incompressible Newtonian viscous fluids**:

$$\rho\,\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} - \mu\,(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\boldsymbol{u} = -\,\boldsymbol{\nabla}p + \rho\,\boldsymbol{g} \qquad (6.53)$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0\,. \qquad (6.54)$$

We first return to **Bernoulli's streamline theorem** for incompressible **inviscid flows**. Writing the acceleration of the conservative body force due to gravity as $\boldsymbol{g} = -\,\boldsymbol{\nabla}\Phi$ (force per unit mass), using the vector identities (the first being valid in **Cartesian coordinates** *only*)

$$(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\boldsymbol{u} = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{u}) - \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{u}) \qquad (6.55)$$

$$(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} = \frac{1}{2}\boldsymbol{\nabla}(\boldsymbol{u}\cdot\boldsymbol{u}) + (\boldsymbol{\nabla}\times\boldsymbol{u})\times\boldsymbol{u} \qquad (6.56)$$

and noting that

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \frac{\partial\boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}\,, \qquad (6.57)$$

while remembering $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{u}$, we can thus write the **Navier–Stokes momentum equations** as

$$\frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{\omega}\times\boldsymbol{u} + \nu\,\boldsymbol{\nabla}\times\boldsymbol{\omega} = -\,\boldsymbol{\nabla}H\,, \qquad (6.58)$$

where $\nu := \mu/\rho$ and the fluid **specific gravito-enthalpy** is defined as $H = \frac{1}{2}u^2 + (p/\rho) + \Phi$. This equation has some simple corrolaries:

1. In **steady** ($\partial\boldsymbol{u}/\partial t = 0$), **inviscid** ($\nu = 0$) motion the fluid specific gravito-enthalpy H is *constant along streamlines*. This can be seen from

$$\boldsymbol{u}\cdot\boldsymbol{\nabla}H = -\boldsymbol{u}\cdot(\boldsymbol{\omega}\times\boldsymbol{u}) = 0\,. \qquad (6.59)$$

2. If the flow is **irrotational** ($\boldsymbol{\omega} = 0$), then

$$\frac{\partial \phi}{\partial t} + H = f(t) \; , \tag{6.60}$$

where $f(t)$ is some function of time. This can be proved as follows. Because the flow is irrotational, a velocity scalar potential can be introduced by $\boldsymbol{u} = \boldsymbol{\nabla}\phi$. Hence, the **Navier–Stokes momentum equations** can be written

$$\boldsymbol{\nabla}\left[\frac{\partial \phi}{\partial t} + H\right] = \mathbf{0} \; . \tag{6.61}$$

Thus, $\frac{\partial \phi}{\partial t} + H$ is independent of the spatial coordinates, so the only possibility is that it is a function of time only (or constant).

3. In a **steady, irrotational flow** (still incompressible and inviscid), the fluid specific gravito-enthalpy H is *constant everywhere* in the fluid, not just on streamlines.

6.7 Hydrostatics

This is the case when the flow velocity is everywhere zero, $\boldsymbol{u} = \mathbf{0}$. Obviously, viscous forces are *zero* then, and the **Navier–Stokes equations** reduce to

$$\boldsymbol{g} = \frac{1}{\rho}\,\boldsymbol{\nabla} p \; . \tag{6.62}$$

This is actually true for incompressible as well as compressible fluids. Thus, all of the previous material on **hydrostatics** applies both to **viscous** and to **inviscid fluids**.

6.8 Aerofoils and lift

If the flow is incompressible and inviscid, then according to **Bernoulli's streamline theorem** the fluid specific gravito-enthalpy given by $H = \frac{1}{2}\,u^2 + (p/\rho) + \Phi$ is constant along streamlines. We have already argued that there is a class of **viscous flows** around objects where viscous effects are confined to a thin region next to the surface of the object — the boundary layer —, and that outside this boundary layer the flow has a large **Reynolds number** (i.e., viscous effects can be neglected). There the flow is approximately inviscid, and this region is called the region of mainstream flow. This describes the situation of aerofoils inclined at small angles to an incident flow.

Now, the immediate consequence of **Bernoulli's streamline theorem** is that, wherever along a streamline in a flow the magnitude of the flow velocity increases, the pressure must decrease. At some level this explains why airplanes fly! As the air flowing over the top surface of a wing must speed up, the pressure must drop, and the wing feels a pressure force in the *upward* direction (which is called **lift**).

In order to have a *net force* on the **aerofoil** there must be a *non-zero* **circulation** around it; and for the force to be a *lift* the circulation must be *negative*. One of the most important results of aerodynamics is the **Kutta–Joukowski lift theorem**, namely, that the lift L for a narrow aerofoil at small angle to a uniform flow U of constant mass density ρ is given by

$$L = -\,\rho\, U \Gamma \ , \tag{6.63}$$

where Γ is the circulation around the wing. Note that this holds for irrotational flow past a two-dimensional object of any shape or size. The lift depends on the object only via the value of the circulation that the object produces when moved. Of course, from **Kelvin's circulation theorem**, we can only *change* the circulation around some curve moving with the fluid via *viscous effects*. And we have to remember that the aerofoil starts from rest. So even though this important result is derived for an inviscid fluid, it *only* works because of **viscosity**! In the present situation, we have the strange effect that what keeps an airplane flying is a circulation around the wing generated when it first started to move — even before it took off!

6.8.1 Forces on objects in inviscid, incompressible and irrotational flows

We can demonstrate how to calculate the forces on objects in inviscid, incompressible, irrotational flows. In this situation, **Bernoulli's streamline theorem** states that the fluid specific (gravito-)enthalpy H is constant everywhere. In the *absence* of body forces (i.e., no buoyancy), we have

$$H = \tfrac{1}{2}\, u^2 + \frac{p}{\rho} = \text{constant} \ . \tag{6.64}$$

If we know the flow velocity $\boldsymbol{u}$ (e.g., from any of our earlier examples), then we can calculate the pressure at the *surface* of the object. Pressure acts normally to the surface, so one can calculate the total force on the object by appropriate integration over its surface.

Force on a sphere in a uniform flow (zero circulation)

For a sphere of radius a, immersed in a uniform flow $\boldsymbol{u} = U\boldsymbol{e}_z$, $U =$ constant, we have the **velocity scalar potential**

$$\phi = Ur\cos\vartheta\left(1 + \frac{a^3}{2r^3}\right) , \tag{6.65}$$

using **spherical polar coordinates**. So, remembering that $u_r = 0$ at $r = a$ (boundary condition for inviscid flow), the only *non-zero* component of the flow velocity *on* the boundary is

$$u_\vartheta = \frac{1}{r}\frac{\partial\phi}{\partial\vartheta} = -U\sin\vartheta\left(1 + \frac{a^3}{2r^3}\right) . \tag{6.66}$$

So, at $r = a$,

$$u_\vartheta = -\frac{3}{2}U\,\sin\vartheta . \tag{6.67}$$

Hence, the pressure at $r = a$ is

$$p = p_0 - \frac{9}{8}U^2\rho\sin^2\vartheta . \tag{6.68}$$

Consider an annulus between ϑ and $\vartheta + \mathrm{d}\vartheta$. The normal force on this is $p2\pi a^2\sin\vartheta\mathrm{d}\vartheta$, and the corresponding force component along the polar z–axis is $p2\pi a^2\sin\vartheta\cos\vartheta\mathrm{d}\vartheta$. Thus, the total force is a **drag force** given by

$$\boldsymbol{F} = 2\pi a^2\int_0^\pi p\sin\vartheta\cos\vartheta\,\mathrm{d}\vartheta\;\boldsymbol{e}_z . \tag{6.69}$$

With (6.68) this becomes

$$\boldsymbol{F} = 2\pi a^2\left[\tfrac{1}{2}p_0\int_0^\pi \sin 2\vartheta\,\mathrm{d}\vartheta - \frac{9}{8}U^2\rho\int_0^\pi \sin^3\vartheta\cos\vartheta\,\mathrm{d}\vartheta\right]\;\boldsymbol{e}_z = \boldsymbol{0} , \tag{6.70}$$

so the **total drag force** is *zero*, and, by symmetry of the pressure, the force in the perpendicular direction (the lift) is also *zero*!

This is known as **d'Alembert's paradox**. It can be generalized to arbitrary shapes as follows:

The force on any body due to an incompressible, irrotational, steady flow, with zero circulation, is zero.

It is not too much of a paradox: If there is *no* circulation, then we would expect *no* flow asymmetry, hence *no* lift.

Force on a cylinder in a uniform flow (non-zero circulation)

Earlier we had found that for an infinitely extended rigid cylinder of radius a, emersed in a uniform flow of strength U, the **velocity scalar potential** was given in **spherical polar coordinates** by

$$\phi = Ur\cos\vartheta\left[1 + \frac{a^2}{r^2}\right] + \frac{\Gamma}{2\pi}\,\vartheta\ . \tag{6.71}$$

We had to add the last term (one of the $m = 0$ parts of the cylindrical harmonics) to account for the non-zero circulation Γ around the cylinder. As in the previous example, let us calculate the velocity component u_ϑ *on* the surface of the cylinder. Generally,

$$u_\vartheta = \frac{1}{r}\,\frac{\partial\phi}{\partial\vartheta} = -\,U\sin\vartheta\left[1 + \frac{a^2}{r^2}\right] + \frac{\Gamma}{2\pi r}\ , \tag{6.72}$$

so that, at $r = a$,

$$u_\vartheta = -\,2U\sin\vartheta + \frac{\Gamma}{2\pi a}\ . \tag{6.73}$$

The inviscid boundary condition means that the normal velocity component is zero on the boundary surface, so this surface must also be a streamline (the fluid velocity is tangential to the boundary surface). By **Bernoulli's streamline theorem** we have that $\frac{1}{2}\,\rho\,u^2 + p = \mathrm{constant}$ at $r = a$, i.e.,

$$p = p_0 - \frac{\rho\Gamma^2}{8\pi^2a^2} - 2\rho U^2\sin^2\vartheta + \frac{\rho U\,\Gamma}{\pi a}\,\sin\vartheta\ . \tag{6.74}$$

Let us now consider a strip of the cylinder between ϑ and $\vartheta + \mathrm{d}\vartheta$, of area $a\mathrm{d}\vartheta$ per unit length of the cylinder. (Everything is now in terms of "per unit length of cylinder".) The **normal force**, which acts radially, is then given by $-pa\mathrm{d}\vartheta$. This force can be decomposed into its x– and y–components; remember that the flow is in the positive x–direction. The **total force** on the cylinder is now expressed by

$$\boldsymbol{F} = -\int_0^{2\pi} pa\cos\vartheta\,\mathrm{d}\vartheta\;\boldsymbol{e}_x - \int_0^{2\pi} pa\sin\vartheta\,\mathrm{d}\vartheta\;\boldsymbol{e}_y\ . \tag{6.75}$$

By considering the symmetries of the functions involved in this expression, one sees that integrating between 0 and 2π produces zero for each of the following:

$\sin\vartheta$, $\cos\vartheta$, $\sin^2\vartheta\cos\vartheta$, $\sin^3\vartheta$, $\sin\vartheta\cos\vartheta$. This only leaves *one* non-zero term:

$$\boldsymbol{F} = -\,\frac{\rho\,U\,\Gamma}{\pi a}\,a\int_0^{2\pi}\sin^2\vartheta\,\mathrm{d}\vartheta\;\boldsymbol{e}_y\;. \tag{6.76}$$

We integrate this, using $\sin^2\vartheta = \frac{1}{2}\,(1 - \cos 2\vartheta)$, to obtain

$$\boldsymbol{F} = -\,\rho\,U\,\Gamma\;\boldsymbol{e}_y\;. \tag{6.77}$$

Thus, surprisingly, there is *no* force in the x–direction, i.e., *no* drag. There *is*, however, a force perpendicular to the flow, i.e., a **lift**, and this lift is proportional to the *negative of the circulation around the cylinder*. In fact this is none other than an illustration of the **Kutta–Joukowski theorem**!

Chapter 7
Incompressible viscous flows

We now turn to examples of **incompressible flow** that is **viscous** of the **Newtonian kind**. Some aspects of viscous flow were illustrated earlier, such as the role of **viscosity** in the **diffusion of vorticity**.

As a reminder, the **Navier–Stokes equations** and the **continuity equation** in this situation are

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} - \nu\,(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\boldsymbol{u} = -\boldsymbol{\nabla}\left(\frac{p}{\rho}\right) + \boldsymbol{g} \qquad (7.1)$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0\,, \qquad (7.2)$$

with the **mass density** assumed as ρ = constant and the **kinematic shear viscosity** defined by $\nu := \mu/\rho$, which is likewise constant. The physical dimension of the latter is $[\,\text{length}\,]^2\,[\,\text{time}\,]^{-1}$. The **boundary conditions** for **viscous flow** are that at a stationary boundary the flow velocity $\boldsymbol{u}$ must be zero. As stated in chapter 3 before this is called the **no-slip boundary condition**. Compare this with the **boundary conditions** for **ideal fluids**, where only the normal component of the flow velocity must be zero at a stationary boundary, so that there can be a non-zero "**slip flow**" tangential to a boundary.

7.1 One-dimensional flow

This is a particular situation where a number of different problems can be solved. In **Cartesian coordinates** one takes

$$\boldsymbol{u} = u_x\,\boldsymbol{e}_x\ .$$

The **incompressibility condition** $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$ implies

$$\frac{\partial u_x}{\partial x} = 0\,, \qquad (7.3)$$

and hence the convective term in the **Navier–Stokes equations**, $(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}$, *drops out*, reducing this partial differential equation to a *linear* subcase. In the *absence* of body forces the **Navier–Stokes equations** can then be written as

$$\frac{\partial u_x}{\partial t} - \nu\left(\frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right) = -\frac{1}{\rho}\,\frac{\partial p}{\partial x} \qquad (7.4)$$

$$0 = \frac{\partial p}{\partial y} = \frac{\partial p}{\partial z}\,. \qquad (7.5)$$

Rearranging the former we get

$$\frac{\partial p}{\partial x} = \mu \left(\frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2} \right) - \rho \, \frac{\partial u_x}{\partial t} \; . \tag{7.6}$$

Now the right-hand side terms contain only u_x, which is *independent* of x as a consequence of $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$. On the other hand, p, and hence the left-hand side term $\partial p / \partial x$, are *independent* of y and z by (7.5). It thus follows that the only remaining allowed dependence of both sides of the equation is on t, so

$$\frac{\partial p}{\partial x} = -\, G(t) \; , \tag{7.7}$$

where $G(t)$ is some function of time.

7.1.1 Steady flow between fixed parallel plates

For **steady flow**, in the positive $\boldsymbol{e}_x$–direction, we must have that the function $G(t)$ is just a *constant*, i.e.,

$$\frac{\partial p}{\partial x} = \mu \, \frac{\partial^2 u_x}{\partial y^2} = -\, k \; . \tag{7.8}$$

In other words, there is a *constant* **pressure gradient** which drives the flow. We have

$$p = p_0 - kx \; , \tag{7.9}$$

so from (7.8)

$$u_x = D + Cy - \frac{ky^2}{2\mu} \; , \tag{7.10}$$

where D and C are arbitrary integration constants.

Now consider parallel plates located at $y = +h$ and $y = -h$. The **boundary condition** for **viscous flow** is the **no-slip condition**, i.e., $u_x = 0$ at $y = \pm h$. Thus we can solve for the constants, obtaining

$$C = 0 \, , \qquad D = \frac{kh^2}{2\mu} \; . \tag{7.11}$$

Hence, the final solution is

$$u_x = \frac{k}{2\mu} \, (h^2 - y^2) \; . \tag{7.12}$$

Note that the flow velocity $\boldsymbol{u}$ is thus proportional to the **pressure gradient** (given by $-k$), and the profile of the flow perpendicular to the flow is *parabolic*. In fact, we had already obtained this result earlier in chapter 3.

7.1.2 Steady flow through circular pipe

Using **cylindrical polar coordinates**, we consider a **steady flow** of the form

$$\boldsymbol{u} = u_z\,\hat{\boldsymbol{e}}_z\ .$$

Assuming **axial symmetry**, i.e., $\partial\boldsymbol{u}/\partial\varphi = \boldsymbol{0}$, and **incompressibility** of the fluid, we find

$$u_z = u_z(r)$$

in order to satisfy $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$. This corresponds to flow down a circular pipe whose axis is aligned with the polar $\hat{\boldsymbol{e}}_z$–axis.

In steady flow, using the appropriate forms of the differential operators, we have

$$\frac{\partial p}{\partial z} = \mu\,\frac{1}{r}\,\frac{\partial}{\partial r}\left(r\,\frac{\partial u_z}{\partial r}\right) \tag{7.13}$$

$$0 = \frac{\partial p}{\partial r} = \frac{\partial p}{\partial \varphi}\ . \tag{7.14}$$

Now the right-hand side of the first equation is a function of r only, and its left-hand side is independent of r; hence, both terms are *constant*, say $-G$. Thus, integrating the right-hand side yields

$$\frac{\partial}{\partial r}\left(r\,\frac{\partial u_z}{\partial r}\right) = -\,\frac{G}{\mu}\,r\ , \tag{7.15}$$

then

$$r\,\frac{\partial u_z}{\partial r} = -\,\frac{G}{2\mu}\,r^2 + A\ , \tag{7.16}$$

and finally

$$u_z = -\,\frac{G}{4\mu}\,r^2 + A\,\ln r + B\ , \tag{7.17}$$

where A and B are integration constants. Integrating the left-hand side yields

$$p = -\,Gz + C\ , \tag{7.18}$$

with C an integration constant. For a **circular pipe** of radius a, the **no-slip boundary condition** is $u_z = 0$ at $r = a$. Moreover, we also want a solution that is *finite* (regular) at $r = 0$. This fixes the integration constants A and B to be

$$A = 0\ , \qquad B = \frac{Ga^2}{4\mu}\ . \tag{7.19}$$

Hence, the final solution is

$$u_z(r) = \frac{G}{4\mu}\,(a^2 - r^2)\ . \tag{7.20}$$

Here again the **flow velocity $\boldsymbol{u}$** is proportional to the **pressure gradient** (which is $-G$), and has a *parabolic* form across a diameter of the pipe.

Once we know how fast the fluid is moving in the tube, we can calculate the **volume flux** through a cross-section S of the pipe:

$$Q = \int\int_S \boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}A = 2\pi\int_0^a r u_z(r)\,\mathrm{d}r = 2\pi\frac{G}{4\mu}\int_0^a (a^2 r - r^3)\,\mathrm{d}r\ , \quad (7.21)$$

so

$$Q = \frac{\pi G a^4}{8\mu}\ . \quad (7.22)$$

This result can be used to *measure* the **viscosity** (here: the value of μ) of a fluid by seeing how Q varies for different applied **pressure gradients**.

7.1.3 Steady flow under gravity down an inclined plane

Let us arrange the axes of a **Cartesian coordinate system** so that the x–axis is along the surface of the **inclined plane** (with angle of inclination α), and the y–axis is perpendicular to the plane. The components of the **gravitational acceleration** are thus given by

$$\boldsymbol{g} = g\,(\sin\alpha, -\cos\alpha, 0)^T\ .$$

The **no-slip boundary condition** means that $\boldsymbol{u} = \boldsymbol{0}$ on $y = 0$.

We might expect that the **flow velocity** $\boldsymbol{u}$ has only a component in the $\boldsymbol{e}_x$–direction, but in the absence of extra information we shall *assume* a form

$$\boldsymbol{u} = [u_x(y), u_y(y), 0]^T\ .$$

Applying the **incompressibility condition**, $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$, we find straightaway that

$$\frac{\mathrm{d}u_y}{\mathrm{d}y} = 0\ , \quad (7.23)$$

i.e., that u_y is constant. But $u_y = 0$ on $y = 0$, so $u_y = 0$ everywhere.

Substituting $\boldsymbol{u} = [u_x(y), 0, 0]^T$ into the **Navier–Stokes equations** gives, for **steady flow**,

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\,\frac{\mathrm{d}^2 u_x}{\mathrm{d}y^2} + g\sin\alpha \quad (7.24)$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} - g\cos\alpha \quad (7.25)$$

$$0 = \frac{\partial p}{\partial z}\ . \quad (7.26)$$

Integrating the second, we get

$$p = -\,\rho g y \cos\alpha + f(x)\ , \tag{7.27}$$

where $f(x)$ is an arbitrary function of x.

Now we consider the **boundary conditions** at the so-called **free surface** (i.e., the upper surface of the flowing fluid). The free surface must be at $y = h$, because all the **streamlines** are parallel to the plane, and the free surface *must* be defined by a set of streamlines. At the free surface *the fluid pressure must equal the atmospheric pressure*, as otherwise there would not be an equilibrium sustained, and then the flow would not be steady. Also, at this free surface, *the tangential stresses must be zero*, since there is *no* fluid above to balance it if it were non-zero. So we have to satisfy the conditions

$$p = p_0\ , \qquad \nu\,\frac{\mathrm{d}u_x}{\mathrm{d}y} = 0\ , \quad \text{at} \quad y = h\ . \tag{7.28}$$

Consequently, $f(x)$ must just be a constant, and the **pressure** thus becomes

$$p - p_0 = \rho g (h - y)\cos\alpha\ . \tag{7.29}$$

Our result clearly shows that $\partial p/\partial x = 0$. This simplifies (7.24) to

$$\nu\,\frac{\mathrm{d}^2 u_x}{\mathrm{d}y^2} = -\,g\sin\alpha\ , \tag{7.30}$$

which we have to solve subject to the conditions

$$u_x = 0 \quad \text{at} \quad y = 0\ , \qquad \nu\,\frac{\mathrm{d}u_x}{\mathrm{d}y} = 0 \quad \text{at} \quad y = h\ . \tag{7.31}$$

Integrating twice we get

$$u_x = -\,\frac{g\sin\alpha}{2\nu}\,y^2 + Ay + B\ , \tag{7.32}$$

where A and B are integration constants. From this we obtain

$$\frac{\mathrm{d}u_x}{\mathrm{d}y} = -\,\frac{g\sin\alpha}{\nu}\,y + A\ . \tag{7.33}$$

Now determining A and B from the **boundary conditions** yields

$$B = 0\ , \qquad A = \frac{g\sin\alpha}{\nu}\,h\ . \tag{7.34}$$

So that the final solution is

$$u_x = \frac{g\sin\alpha}{2\nu}\,(2h - y)\,y\ . \tag{7.35}$$

Again, the velocity profile is *parabolic*. Also, as before, we can find the **volume flux** (now per unit length in $\boldsymbol{e}_z$–direction) down the plane, which is

$$Q = \int_0^h u_x\,\mathrm{d}y = \frac{g h^3}{3\nu}\,\sin\alpha\ . \tag{7.36}$$

7.1.4 Flow due to an impulsively moved plane boundary

We consider a **viscous fluid** which is at rest in the region $0 < y < \infty$ of a **Cartesian coordinate system**. At time $t = 0$ the **plane rigid boundary**, which we assume to hold the fluid at $y = 0$, is suddenly put into motion at a *constant* speed U in the $\boldsymbol{e}_x$–direction. The **no-slip boundary condition** means that the fluid at $y = 0$ must move with the *same* velocity as the boundary. We expect that the effect of this motion is that gradually the rest of the fluid, away from the boundary, will also gain a velocity in the $\boldsymbol{e}_x$–direction.

Because of the **symmetry** in the present configuration, we expect the flow to be in the $\boldsymbol{e}_x$–direction, but, in view of (7.3), to depend only on the coordinates t and y. This corresponds to a **plane parallel shear flow** with **flow velocity**

$$\boldsymbol{u} = u_x(t, y)\, \boldsymbol{e}_x \ . \tag{7.37}$$

The corresponding **Navier–Stokes equations** for an **incompressible Newtonian viscous fluid** with $\rho =$ constant are

$$\frac{\partial u_x}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\,\frac{\partial^2 u_x}{\partial y^2} \tag{7.38}$$

$$0 = \frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} \ , \tag{7.39}$$

which in the present case have reduced again to a set of partial differential equations which are *linear*. By (7.39) the **pressure** can be a function of t and x only. On the other hand, by (7.38), $\partial p/\partial x$ is the difference of two terms that each are *independent* of x; thus $\partial p/\partial x$ can be a function of t only, say $f(t)$. Integrating we then obtain

$$p(t, x) = f(t)x + p_0 \ ,$$

with $p_0 =$ constant. However, in the present case it is reasonable to *assume* that the pressure at $x = \pm\infty$ is the *same*, for all values of t (i.e., no external driving pressures active). Clearly this condition leads to

$$f(t) = 0 \qquad \Rightarrow \qquad p = p_0 \ . \tag{7.40}$$

Thus, the *partial* differential equation governing the time evolution of the **flow velocity** component $u_x(t, y)$ is the classical **diffusion equation**,

$$\boxed{\frac{\partial u_x}{\partial t} - \nu\,\frac{\partial^2 u_x}{\partial y^2} = 0 \ ,} \tag{7.41}$$

here in its 1-D realisation. We need to solve the **diffusion equation** subject to the **initial condition**

$$u_x(t = 0, y) = 0 \quad \text{for} \quad y > 0 \ , \tag{7.42}$$

and the **boundary conditions**

$$u_x(t, y = 0) = U \quad \text{for} \quad t > 0 \ , \qquad u_x(t, y \to \infty) = 0 \quad \text{for} \quad t > 0 \ . \tag{7.43}$$

A *successful strategy* to achieve this is to look for a so-called **similarity solution** to the **diffusion equation**. Note that the **diffusion equation** is **invariant** (unchanged) under a **re-scaling** of the independent variables t and y according to

$$t \to \alpha^2 t \;, \qquad y \to \alpha y \;, \tag{7.44}$$

where α is a constant. This suggests that there may be **special solutions** of the **diffusion equation** which are functions of t and y in the particular combination $y/t^{1/2}$ (since in this case the α of the above variable transformation cancels out immediately). So let us try the **Ansatz**

$$u_x = U\,f(\eta) \;, \quad \text{with} \quad \eta := \frac{y}{(\nu t)^{1/2}} \;. \tag{7.45}$$

We emphasise that the *new* independent variable η thus defined is **dimensionless**,[1] as is the function $f = f(\eta)$. It is a straightforward mathematical exercise to show that by the chain rule of differentiation the partial derivatives occurring in (7.41) have to be transformed according to

$$\frac{\partial u_x}{\partial t} = U f'(\eta)\,\frac{\partial \eta}{\partial t} = -\,U f'(\eta)\,\frac{y}{2\nu^{1/2}t^{3/2}} = -\,U f'(\eta)\,\frac{\eta}{2t} \tag{7.46}$$

$$\frac{\partial u_x}{\partial y} = U f'(\eta)\,\frac{\partial \eta}{\partial y} = U f'(\eta)\,\frac{1}{(\nu t)^{1/2}} \tag{7.47}$$

$$\frac{\partial^2 u_x}{\partial y^2} = U f''(\eta)\,\frac{1}{(\nu t)} \;. \tag{7.48}$$

Upon substitution this converts the **diffusion equation** (7.41) to

$$\nu\,U f''(\eta)\,\frac{1}{(\nu t)} + U f'(\eta)\,\frac{\eta}{2t} = 0 \;. \tag{7.49}$$

Therefore, the *ordinary* differential equation to be satisfied by the function $f = f(\eta)$ is given by

$$f'' + \frac{1}{2}\,\eta f' = 0 \;, \tag{7.50}$$

with a prime denoting a derivative with respect to η. Integrating this equation once yields

$$f' = B\,\mathrm{e}^{-\eta^2/4} \;, \tag{7.51}$$

with B an integration constant. Integrating once again then leads to

$$f(\eta) = A + B\int_0^{\eta} \mathrm{e}^{-s^2/4}\,\mathrm{d}s \;, \tag{7.52}$$

with A a second integration constant. In deriving this result we made use of the **integral identity** $\int_a^b g'(x)\,\mathrm{d}x = g(b) - g(a)$. The constants A and B are determined from the **boundary** and **initial conditions** as follows. From (7.42) and (7.43) we have that

$$f(\eta \to \infty) = 0 \;, \qquad f(\eta = 0) = 1 \;. \tag{7.53}$$

[1]The kinematical viscosity, defined as $\nu := \mu/\rho$, has physical dimension $[\,\text{length}\,]^2\,[\,\text{time}\,]^{-1}$. Thus, νt has physical dimension $[\,\text{length}\,]^2$, and $y/(\nu t)^{1/2}$ is dimensionless.

Hence, our final solution for the **flow velocity** component u_x becomes

$$u_x(t,y) = u_x(\eta) = U \left[1 - \frac{1}{\sqrt{\pi}} \int_0^\eta \mathrm{e}^{-s^2/4}\, \mathrm{d}s \right] , \tag{7.54}$$

remembering that $\eta = y/(\nu t)^{1/2}$. One obtains this result by noting that

$$\int_0^\infty \mathrm{e}^{-\alpha x^2}\, \mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} . \tag{7.55}$$

The velocity profiles described by the solution (7.54) have the *same* **shape** (i.e., they exhibit a "similarity"), but are *increasingly stretched out* in the $\boldsymbol{e}_y$–direction as time proceeds. The dynamical effects on the fluid by the motion of the plane boundary are confined to a region within about a **distance** $(\nu t)^{1/2}$ of the moving boundary. The **vorticity** in the fluid motion gradually diffuses away from the moving boundary. We find that it is given by

$$\boldsymbol{\omega} = \omega_z\, \boldsymbol{e}_z = -\frac{\partial u_x}{\partial y}\, \boldsymbol{e}_z = \frac{U}{(\pi \nu t)^{1/2}}\, \mathrm{e}^{-y^2/4\nu t}\, \boldsymbol{e}_z . \tag{7.56}$$

Clearly the effect of **viscosity** is to increasingly *smooth out* what was initially a **vortex sheet** at the moving boundary.

7.2 Flow with circular streamlines

As a slightly more complex example of **viscous flow**, we now turn to the situation where the **streamlines** are assumed to be **circular**, e.g., in situations where there is **cylindrical symmetry**.

We start from the **Navier–Stokes equations** for **incompressible Newtonian viscous flow**, given by (in the absence of gravity)

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} - \nu\,(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\boldsymbol{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p \tag{7.57}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0 . \tag{7.58}$$

Since the streamlines are to be circular, we can assume the following form for the **flow velocity**:

$$\boldsymbol{u} = u_\varphi(t,r)\, \hat{\boldsymbol{e}}_\varphi . \tag{7.59}$$

We can check that this automatically satisfies the **incompressibility condition**, writing the divergence operator in the form for **cylindrical polar coordinates**:

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = \underbrace{\frac{1}{r}\frac{\partial}{\partial r}(r\, u_r)}_{u_r = 0} + \frac{1}{r}\frac{\partial u_\varphi}{\partial \varphi} + \underbrace{\frac{\partial u_z}{\partial z}}_{u_z = 0} = 0 . \tag{7.60}$$

Next, we write out the **Navier–Stokes equations** in component form, and then aim to eliminate p, so that we will gain an equation just for u_φ. But there are dangers! The expressions for $(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}$ and $(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\boldsymbol{u}$ in **cylindrical polar coordinates** are *not* straightforward because the unit basis vectors $\hat{\boldsymbol{e}}_r$ and $\hat{\boldsymbol{e}}_\varphi$

are themselves functions of φ. There are *different ways* to treat this problem. One could use standard expressions as given in textbooks, or one could use the following relationships:

$$\frac{\partial \hat{\boldsymbol{e}}_r}{\partial \varphi} = \hat{\boldsymbol{e}}_\varphi \, , \qquad \frac{\partial \hat{\boldsymbol{e}}_\varphi}{\partial \varphi} = -\,\hat{\boldsymbol{e}}_r \, , \qquad \frac{\partial \hat{\boldsymbol{e}}_z}{\partial \varphi} = \boldsymbol{0} \, , \tag{7.61}$$

or one could use standard vector identities to expand out the troublesome terms. Here we use a mixture of the three approaches!

From chapter 2 we have

$$\boldsymbol{\nabla} p = \left(\frac{\partial p}{\partial r}, \, \frac{1}{r}\frac{\partial p}{\partial \varphi}, \, \frac{\partial p}{\partial z} \right)^T \; . \tag{7.62}$$

Also we can consult a reference book to find

$$(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} = \left[(\boldsymbol{u}\cdot\boldsymbol{\nabla})u_r - \frac{u_\varphi^2}{r}, \, (\boldsymbol{u}\cdot\boldsymbol{\nabla})u_\varphi + \frac{u_r u_\varphi}{r}, \, (\boldsymbol{u}\cdot\boldsymbol{\nabla})u_z \right]^T \; . \tag{7.63}$$

Remembering that we presently have $u_r = u_z = 0$, we concentrate on the term

$$(\boldsymbol{u}\cdot\boldsymbol{\nabla})u_\varphi = \left(u_r \frac{\partial}{\partial r} + \frac{u_\varphi}{r}\frac{\partial}{\partial \varphi} + u_z \frac{\partial}{\partial z} \right) u_\varphi = 0 \, . \tag{7.64}$$

Thus this only leaves *one* non-zero term (!), given by

$$(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} = \left(\frac{-\,u_\varphi^2}{r}, \, 0, \, 0 \right)^T \; . \tag{7.65}$$

For the $(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\boldsymbol{u}$ term we will use the vector identity

$$(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\boldsymbol{u} = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{u}) - \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{u}) \; . \tag{7.66}$$

In our example the first term on right-hand side is zero, and for the curl of $\boldsymbol{u}$ in **cylindrical polar coordinates** we use the form

$$\boldsymbol{\nabla}\times\boldsymbol{u} = \frac{1}{r} \begin{vmatrix} \hat{\boldsymbol{e}}_r & r\,\hat{\boldsymbol{e}}_\varphi & \hat{\boldsymbol{e}}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ 0 & r\,u_\varphi & 0 \end{vmatrix} = \frac{1}{r}\frac{\partial}{\partial r}(r\,u_\varphi)\,\hat{\boldsymbol{e}}_z \, . \tag{7.67}$$

And then again …

$$-\,\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{u}) = -\,\frac{1}{r} \begin{vmatrix} \hat{\boldsymbol{e}}_r & r\,\hat{\boldsymbol{e}}_\varphi & \hat{\boldsymbol{e}}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ 0 & 0 & \frac{1}{r}\frac{\partial}{\partial r}(r\,u_\varphi) \end{vmatrix} \tag{7.68}$$

$$= -\,\frac{1}{r}(-r)\,\hat{\boldsymbol{e}}_\varphi \frac{\partial}{\partial r}\left[\frac{1}{r}\left(r\frac{\partial u_\varphi}{\partial r} + u_\varphi \right) \right] \; . \tag{7.69}$$

Therefore

$$(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\boldsymbol{u} = \left(\frac{\partial^2 u_\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r^2} \right) \hat{\boldsymbol{e}}_\varphi \, . \tag{7.70}$$

With these results we can finally write down the **Navier–Stokes equations**, component by component:

$$-\frac{u_\varphi^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \tag{7.71}$$

$$\frac{\partial u_\varphi}{\partial t} = -\frac{1}{\rho r}\frac{\partial p}{\partial \varphi} + \nu\left(\frac{\partial^2 u_\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r^2}\right) \tag{7.72}$$

$$0 = \frac{\partial p}{\partial z}\,. \tag{7.73}$$

Equations (7.72) and (7.73) lead to the statement that for the pressure p we have

$$\frac{\partial p}{\partial \varphi} = P(t,r) \qquad \Rightarrow \qquad p = P(t,r)\,\varphi + f(t,r)\,, \tag{7.74}$$

where $P(t,r)$ is a *known* function of t and r, and $f(t,r)$ is an *arbitrary* function. Because p must be a *single-valued* function of φ, it follows that $P \equiv 0$, and therefore

$$\frac{\partial p}{\partial \varphi} = 0\,. \tag{7.75}$$

Thus (7.72) becomes

$$\frac{\partial u_\varphi}{\partial t} = \nu\left(\frac{\partial^2 u_\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r^2}\right)\,. \tag{7.76}$$

This is the **governing equation** for u_φ, which must be solved given the appropriate **initial** and **boundary conditions**. For example, if there is a **cylindrical flow** through a **cylinder**, then the relative velocity (u_φ–component) between the fluid and the cylinder's surface must be *zero* according to the **no-slip boundary condition**.

7.2.1 Viscous decay of a line vortex

As an example of the above theory, we study the effects of **viscosity** on a **line vortex**. That is, we take as an **initial condition** the flow pattern of a $1/r$–**vortex** given by

$$\boldsymbol{u} = \frac{\Gamma_0}{2\pi r}\,\hat{\boldsymbol{e}}_\varphi\,, \tag{7.77}$$

where Γ_0 is a constant. This is a solution for a **vortex** in an **inviscid fluid**, with zero **vorticity** except at $r = 0$ where it is infinite. There is, however, a *finite* **circulation** around the origin. The idea is that we can see the *effect of viscosity* on such vortices by using the inviscid solution as an **initial condition** for the **Navier–Stokes equations**.

Rather than working directly with the **flow velocity** $\boldsymbol{u}$, we will use a variable which is just the **circulation** at **radius** r, i.e.

$$\Gamma(t,r) = 2\pi r\,u_\varphi(t,r)\,. \tag{7.78}$$

In terms of this variable the governing equation (7.76) becomes the partial differential equation

$$\frac{\partial \Gamma}{\partial t} = \nu\left(\frac{\partial^2 \Gamma}{\partial r^2} - \frac{1}{r}\frac{\partial \Gamma}{\partial r}\right)\,. \tag{7.79}$$

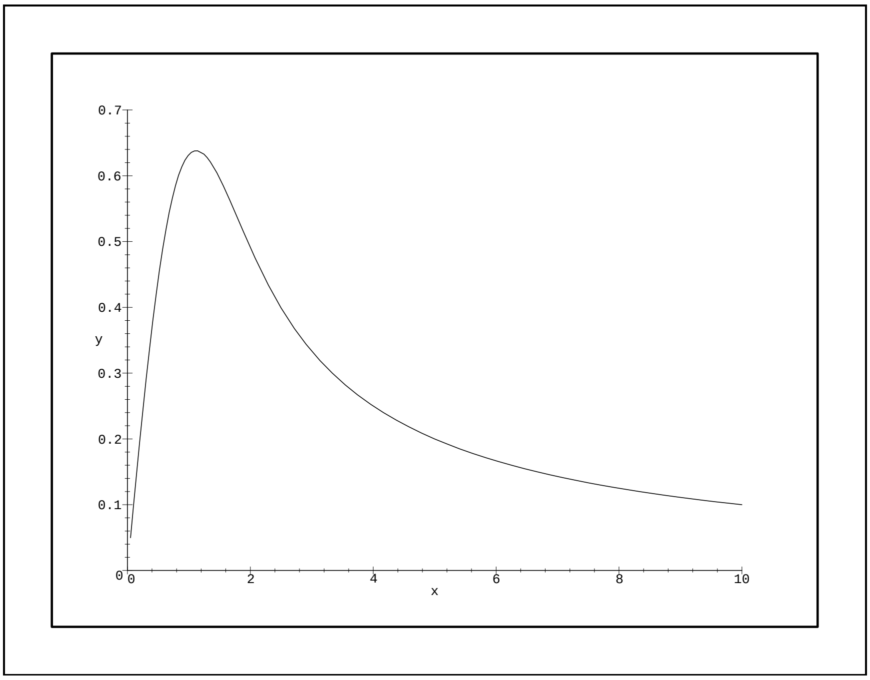
This equation can again be solved with a **similarity solution method**, just as we did in section 7.1.4. One seeks **special solutions** of the form

$$\Gamma = \Gamma_0\, f(\eta)\ , \qquad \text{where} \qquad \eta := \frac{r}{(\nu t)^{1/2}}\ . \tag{7.80}$$

This eventually leads to a solution given by

$$u_\varphi(t,r) = \frac{\Gamma_0}{2\pi r}\left(1 - \mathrm{e}^{-r^2/4\nu t}\right)\ . \tag{7.81}$$

The first term is just the inviscid solution. We see that at fixed r, as time increases, the **flow velocity** component u_φ decreases and departs from a $1/r$-dependence. In other words, the **vorticity** increases. Close to the axis, but within a **distance** that increases in time [i.e., $r \ll (4\nu t)^{1/2}$], the flow is approximately that of uniform rotation. The intensity of the vortex decreases in time, as the core spreads outwards.



Graph of the function $\frac{1}{x}(1 - \mathrm{e}^{-x^2})$*, cf. (7.81).*

Chapter 8
Waves in fluids

8.1 Describing waves

In this section we give a brief introduction to the terminology used in describing **waves**, and introduce in particular the concepts of **group speed** and **dispersion**, which are vital to the understanding of **waves in fluids**.

8.1.1 Terminology

What is a wave?

Generally **wave phenomena** in **physics** are associated with the transport of **energy** without the transport of a material medium. Often we think of a **wave** as a **disturbance**, usually oscillating, around some **equilibrium configuration**. In the present course we will mostly discuss the theory of **small-amplitude sinusoidal disturbances**.

Stationary wave

A **stationary wave** has spatial variation, but *no* time variation. For example, a **sinusoidal stationary wave** is represented by

$$A \cos kx \ , \tag{8.1}$$

where A is the **amplitude** of the wave, and k its **wave number**, the latter having physical dimension $[\,\text{length}\,]^{-1}$. The wave number specifies how rapidly the wave *oscillates in space*. It is related to the wave's **wavelength**, λ, which is the distance between adjacent positions in the wave which differ in **phase** (i.e., the argument to the cosine function) by 2π. For example, the **wave crests** (or the **wave troughs**) have a spacing of one wavelength. Thus

$$\lambda = \frac{2\pi}{k} \ . \tag{8.2}$$

Travelling wave

A **travelling wave** has a spatial variation which is wavelike, *together* with a temporal variation that is also wavelike. Both variations appear in such a way that the profile of the wave moves in space as time increases. For example, a

sinusoidal travelling wave, of **amplitude** A and **wave number** k, is represented by

$$A\cos(kx-\omega t)\ , \tag{8.3}$$

where ω is the **angular frequency** of the wave (usually just called the frequency), which has physical dimension $[\,\text{time}\,]^{-1}$. The angular frequency specifies how rapidly the wave *oscillates in time*. One can see that, as time increases, the **phase** $(kx-\omega t)$ of the wave (i.e., the argument to the cosine function) shifts to lower values, so that for *fixed* x this corresponds to the wave shifting to the *right* (increasing x). Thus the function $A\cos(kx-\omega t)$ represents a wave travelling towards increasing x, for $\omega > 0$. A wave travelling to the *left* (decreasing x) could be obtained by having $\omega < 0$, or choosing the function $A\cos(kx+\omega t)$ with $\omega > 0$.

The SI unit of ω is radians per second (1 rad s^{-1}). We also define a **wave frequency**, f, in terms of the number of complete oscillations per second, i.e.,

$$f=\frac{\omega}{2\pi}\ . \tag{8.4}$$

The SI unit of f is cycles per second (1 s^{-1}), which has been given the name 1 Hz after the German physicist Heinrich Hertz (1857–1894). Given a wave frequency one can calculate how long it takes for a wave to execute one oscillation. This is known as the **wave period**, τ, and we define

$$\tau=\frac{1}{f}=\frac{2\pi}{\omega}\ . \tag{8.5}$$

Standing wave

A **standing wave** oscillates in time and space, but the wave crests do *not* move. For example, a **sinusoidal standing wave** is represented by

$$A\cos kx\ \cos\omega t=\frac{A}{2}\cos(kx-\omega t)+\frac{A}{2}\cos(kx+\omega t)\ . \tag{8.6}$$

As can easily be seen from the given mathematical expression, this is equivalent to the **superposition** of two **travelling waves** with amplitude $A/2$ and angular frequency ω, i.e., one wave propagating forward and one wave propagating backward.

Phase speed

In the case of **travelling waves** one can ask the question of *how fast* a particular part of the wave (such as a wave crest) travels. For example, a wave must move one wavelength in one wave period, so we can define the **phase speed**, c, also known as **wave speed**, by

$$c:=\frac{\lambda}{\tau}=\frac{2\pi/k}{2\pi/\omega}=\frac{\omega}{k}\ . \tag{8.7}$$

This relationship can also be read off from the phase of a **sinusoidal travelling wave**, $k[x-(\omega/k)t]$. This makes clear that the **phase speed** is the rate at which the phase of the wave changes in time.

8.1.2 Group speed and dispersion relation

The **group speed** is the speed of propagation of a group of waves, e.g., a **wave packet**. One could imagine making a wave packet by slowly turning up, then down, the amplitude of a sinusoidal wave generator. One can then describe the wave packet by its **envelope** (i.e., the profile of the averaged wave amplitudes). If this wave packet propagates, then one can measure its speed of propagation, and this is the **group speed**. One uses the group speed because **Fourier analysis** of the wave packet would show components at a number of frequencies: *high frequencies* corresponding to the variation of the **wave phase**, and *low frequencies* corresponding to the variation of the **wave amplitude**.

The **group speed**, c_g, is defined by

$$c_g := \frac{\mathrm{d}\omega}{\mathrm{d}k} \,, \tag{8.8}$$

where $\omega = \omega(k)$ is a medium specific function giving the angular frequency of a wave in terms of its wave number. If the **phase speed** is *not* constant (alternatively, if the phase speed is *not* equal to the group speed), then we say that the medium is **dispersive**. In different terms: for **dispersive media** the *phase speed changes with wave number*.

Thus, when the function $\omega(k)$ is plotted against k, there are three possibilities:

(i) the phase speed c is *constant*, i.e., there is *no* dispersion, so $\omega(k) = ck$;

(ii) the phase speed c *increases* with k; and

(iii) the phase speed c *decreases* with k.

Different examples of dispersion behaviour.

Dispersion relation

The equation

$$\omega = \omega(k) \,, \tag{8.9}$$

where $\omega(k)$ is some function of the wave number k (and possibly also other characteristic parameters of the medium), is known as the **dispersion relation**.

As we shall see shortly, **surface gravity waves** on *deep water* have the following dispersion relation:

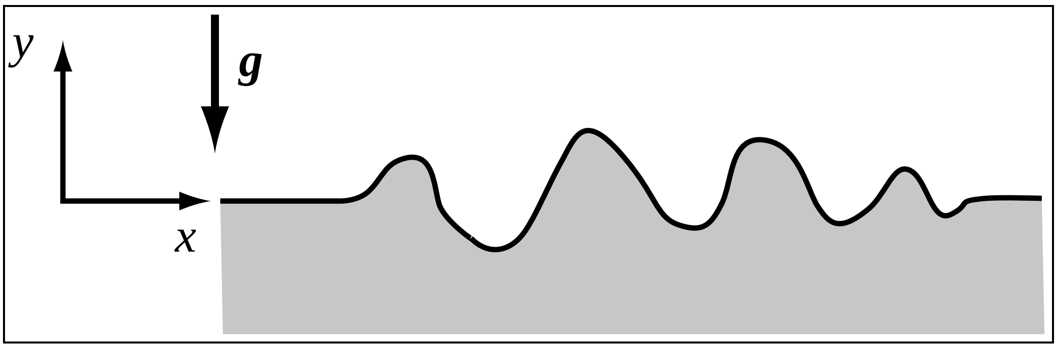
$$\omega(k) = (gk)^{1/2} \quad \Rightarrow \quad c = \left(\frac{g}{k}\right)^{1/2} , \quad c_g = \tfrac{1}{2}\left(\frac{g}{k}\right)^{1/2} = \tfrac{1}{2}\,c\ . \quad (8.10)$$

Thus, the **group speed** is half the **phase speed**, so that if one has a general disturbance, the wave crests within it will move faster than the overall disturbance. In other words, the wave crests will move forward relative to the overall disturbance. This interesting result can be summarised as follows: wave crests appear at the back of a group of waves, make their way to the front, and then disappear. This might seem as rather weird, but careful observation at some pond or lake should *confirm* the result!

The **dispersive effects** of **surface waves** on water are *fundamental* to explaining the complicated wave patterns that are observed in reality, e.g., in the **wake** behind a ship, or the pattern of **ripples** around a fishing line.

8.2 Surface gravity waves on deep water

We examine 2-D waves on deep water, i.e., oscillations of the **free surface** that is the **boundary** between the **water** and the **atmosphere**. This configuration is shown in the figure.



In a **Cartesian coordinate system**, the **flow velocity** takes the form

$$\boldsymbol{u} = u_x(t,x,y)\,\boldsymbol{e}_x + u_y(t,x,y)\,\boldsymbol{e}_y\ . \quad (8.11)$$

Let us suppose that the fluid motion is **irrotational** so that $\boldsymbol{\nabla} \times \boldsymbol{u} = \boldsymbol{0}$. This leads to the condition

$$\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} = 0\ . \quad (8.12)$$

Hence, we can suppose the existence of a **velocity scalar potential** $\phi = \phi(t,x,y)$ such that

$$u_x = \frac{\partial \phi}{\partial x}\ , \qquad u_y = \frac{\partial \phi}{\partial y}\ . \quad (8.13)$$

Assuming in addition **incompressibility** of the water, i.e., $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$, means that the **velocity scalar potential** satisfies **Laplace's equation**:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \; . \tag{8.14}$$

Finally, we have to specify a quantity that actually displays wave motion. In the present case we choose this to be the **height** of the **free surface**, denoted by

$$y = \eta(t, x) \; . \tag{8.15}$$

8.2.1 Surface conditions and linearisation

Kinematic condition at free surface

In order to solve for the wave motion, we must find some conditions which hold at the **free surface**. The first is known as the **kinematic condition** and follows from the statement: *fluid particles on the surface must remain on the surface*. This holds because of the assumption of two-dimensionality. In order for a surface fluid element to exchange positions with a subsurface fluid element (at one instant in time), the first must have a velocity downwards, and the latter must have a velocity upwards. But (because of 2-D) this has to happen at the same point in space. This is obviously impossible. So it is true that surface fluid elements always remain surface fluid elements.

Defining the quantity

$$F(t, x, y) := y - \eta(t, x) \; , \tag{8.16}$$

then for fluid elements on the surface F is zero. And since fluid elements on the surface remain on the surface, F must remain zero on the surface. In other words, the rate of change of F, moving with a fluid element on the surface, is zero. That is, the **convective derivative** of F is zero on the **free surface**:

$$\frac{\mathrm{D}F}{\mathrm{D}t} = \frac{\partial F}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})F = 0 \quad \text{on} \quad y = \eta(t, x) \; . \tag{8.17}$$

Now, using the results

$$\frac{\partial F}{\partial t} = -\frac{\partial \eta}{\partial t} \; , \quad \frac{\partial F}{\partial x} = -\frac{\partial \eta}{\partial x} \; , \quad \frac{\partial F}{\partial y} = 1 \; , \tag{8.18}$$

one finds the **kinematic condition** at the **free surface**:

$$\frac{\partial \eta}{\partial t} + u_x \frac{\partial \eta}{\partial x} = u_y \quad \text{on} \quad y = \eta(t, x) \; . \tag{8.19}$$

One can check if this makes sense. If the free surface is **horizontal**, we have $\partial\eta/\partial x = 0$ so that $\partial\eta/\partial t = u_y$. That is, the rate of change of the height of the free surface is just the y–component of the **flow velocity** $\boldsymbol{u}$ at the free surface. If the free surface is **stationary**, then $\partial\eta/\partial t = 0$ and $\partial\eta/\partial x = u_y/u_x$. That is, the slope of the free surface equals the slope of the streamline at the surface, so that the free surface coincides with a **streamline**. This is correct, since, for the stationary case, particle paths coincide with streamlines.

Pressure condition at free surface

For **unsteady irrotational flow** one can write down a generalised version of **Bernoulli's streamline theorem**. This reads

$$\frac{\partial \phi}{\partial t} + \tfrac{1}{2}\, u^2 + \frac{p}{\rho} + gy = G(t) \;, \tag{8.20}$$

where $G(t)$ is some arbitrary function of time.

Assuming in the present example that the water is **inviscid**,[1] its **pressure** p at the free surface must equal the **atmospheric pressure**, say p_0, which can be assumed to be *constant*. One then chooses the function $G(t)$ such that it absorbs the constant term p_0/ρ and so one finds the **pressure condition** at the **free surface**:

$$\frac{\partial \phi}{\partial t} + \tfrac{1}{2}\,(u_x^2 + u_y^2) + gy = 0 \quad \text{on} \quad y = \eta(t,x) \;. \tag{8.21}$$

Linearisation of surface conditions

We *assume* that the quantities $\eta(t,x)$, u_x and u_y are all "*small*" (i.e., take values near the equilibrium state $y = 0$ and $0 = u_x = u_y$), so that we can **linearise** the **surface conditions**. This means we shall *drop* any terms which are of *higher order than linear* in any of these small quantities.

The **kinematic condition** (8.19) becomes, after dropping the obviously quadratic term and expanding u_y around the **equilibrium position** $y = 0$,

$$u_y(t,x,\eta) = u_y(t,x,0) + \eta \left.\frac{\partial u_y}{\partial y}\right|_{(t,x,0)} + \ldots = \frac{\partial \eta}{\partial t} \;. \tag{8.22}$$

Retaining **linear terms** and using $\boldsymbol{u} = \boldsymbol{\nabla}\phi$, we then get

$$u_y = \frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} \quad \text{on} \quad y = 0 \;. \tag{8.23}$$

One sees here a crucial aspect of the **linearisation procedure**: namely, that the **kinematic condition** is now evaluated at the **equilibrium position** $y = 0$, instead at the position of the **free surface**, the latter of which is, of course, *not yet known*!

One can undertake a similar **linearisation procedure** for the **pressure condition** (8.21). This leads to

$$\frac{\partial \phi}{\partial t} + g\eta = 0 \quad \text{on} \quad y = 0 \;. \tag{8.24}$$

[1]Remember that in this case mechanical pressure and thermodynamical pressure are identical, $P \equiv p$.

8.2.2 Dispersion relation

We now look for a **travelling wave solution** for the **velocity scalar potential** ϕ which *simultaneously satisfies* **Laplace's equation** and the **surface conditions**. We want to *assume* that the y–position of the free surface varies **sinusoidally** according to

$$\eta = A\cos(kx - \omega t) \; . \tag{8.25}$$

This suggests that we look for a ϕ of the form

$$\phi = f(y)\, \sin(kx - \omega t) \; , \tag{8.26}$$

such that also **Laplace's equation**,

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \; , \tag{8.27}$$

is satisfied. Substituting (8.26), we find for $f(y)$ that

$$\frac{\mathrm{d}^2 f}{\mathrm{d}y^2} - k^2 f = 0 \qquad \Rightarrow \qquad f = C\mathrm{e}^{ky} + D\mathrm{e}^{-ky} \; , \tag{8.28}$$

where C and D are integration constants. Without loss of generality we can choose $k > 0$, and then, since we want to *assume* water of "*infinite depth*", we must have $D = 0$, so that the **flow velocity** $\boldsymbol{u}$ remains *bounded* in the limit as $y \to -\infty$. Thus, we get

$$\phi = C\mathrm{e}^{ky}\sin(kx - \omega t) \; . \tag{8.29}$$

Substituting this expression into the **kinematic condition** (8.23) (and remembering that $\mathrm{e}^{ky} = 1$ at $y = 0$) gives

$$Ck = A\omega \; . \tag{8.30}$$

On the other hand, substituting into the **pressure condition** (8.24) gives

$$C\omega = gA \; . \tag{8.31}$$

These two relations can be combined to yield (i)

$$C = \frac{A\omega}{k} \; ,$$

so that our final solution for the **velocity scalar potential** becomes

$$\phi = \frac{A\omega}{k}\,\mathrm{e}^{ky}\, \sin(kx - \omega t) \; . \tag{8.32}$$

In addition, (8.30) and (8.31) can be combined to yield, most importantly, (ii) the **dispersion relation** for **surface gravity waves**, given by

$$\omega^2 = gk \qquad \Rightarrow \qquad \omega(k) = (gk)^{1/2} \; . \tag{8.33}$$

As stated above, this dispersion relation leads to a **phase speed**

$$c = \frac{\omega}{k} = \left(\frac{g}{k}\right)^{1/2} \tag{8.34}$$

and a **group speed**

$$c_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \tfrac{1}{2}\left(\frac{g}{k}\right)^{1/2} = \tfrac{1}{2}\, c\,. \tag{8.35}$$

These are the results discussed in the previous section, which we summarised by the statement: *wave crests appear at the back of a group of waves, make their way to the front, and then disappear*.

Assumptions for linearisation

The above analysis is valid for **small amplitude waves**. From the solution for ϕ one finds that $u^2 \sim A^2\omega^2 = A^2 gk$, so this is the **order of magnitude** of the term *neglected* in **Bernoulli's equation**, (8.20). This approximation is valid if u^2 is *much less than* $g\eta$, i.e., if Ak is small. The latter is equivalent to the statement that the magnitude of the **amplitude** needs to be *much less* than the **wavelength**, i.e.,

$$A \ll \lambda\,.$$

8.2.3 Particle paths in water waves on deep water

The components of the **flow velocity** can be found from $\boldsymbol{u} = \boldsymbol{\nabla}\phi$, so

$$u_x = A\omega\,\mathrm{e}^{ky}\cos(kx - \omega t)\,, \qquad u_y = A\omega\mathrm{e}^{ky}\sin(kx-\omega t)\,. \tag{8.36}$$

Assuming that the particles only depart by some *small* amount, (x', y'), from their **mean positions**, $(\bar{x}, \bar{y})$, the **particle equations of motion** we need to integrate to determine x' and y' and their solutions are given by

$$\frac{\mathrm{d}x'}{\mathrm{d}t} = A\omega\,\mathrm{e}^{k\bar{y}}\cos(k\bar{x}-\omega t) \;\Rightarrow\; x' = -\,A\mathrm{e}^{k\bar{y}}\sin(k\bar{x}-\omega t) \tag{8.37}$$

$$\frac{\mathrm{d}y'}{\mathrm{d}t} = A\omega\,\mathrm{e}^{k\bar{y}}\sin(k\bar{x}-\omega t) \;\Rightarrow\; y' = A\mathrm{e}^{k\bar{y}}\cos(k\bar{x}-\omega t)\,. \tag{8.38}$$

It can be seen that these equations describe uniform **circular motion**, and that the **particle paths** are circular with clockwise sense of orientation. The **radius** of the circles is $A\mathrm{e}^{k\bar{y}}$ and decreases exponentially with depth. Therefore, most of the motion is confined to a region extending about half a wavelength below the water surface. The **flow velocity** at any depth is constant but rotating. The **phase** of the circular motion is the same at all depths for fixed x–position.

8.3 Dispersion and group speed

We have introduced the **group speed** defined as

$$c_g := \frac{\mathrm{d}\omega(k)}{\mathrm{d}k}\,. \tag{8.39}$$

The **group speed** has two important properties:

(i) it is the speed at which an *isolated wave packet travels*;

(ii) it is the speed at which *energy is transported* by waves with wavenumber k.

We will now demonstrate the first property.

Propagation of a wave packet

We consider a **wave packet** with many oscillations of some characteristic wave number, and with a *slowly* spatially varying **envelope** of amplitudes. We can use a **Fourier integral representation** in *space* for any disturbance $\eta(t,x)$, i.e., (cf. chapter 2)

$$\eta(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k)\, \mathrm{e}^{i(kx-\omega t)}\, \mathrm{d}k \ , \tag{8.40}$$

(taking the real part of the right-hand side is understood), where $A(k)$ is a function whose absolute value gives a measure of the amount of wave power at wave number k present in the wave packet. The function $A(k)$ can be calculated from the given function $\eta(t,x)$ according to the general procedure discussed in chapter 2.

We now turn to the case we are interested in, namely, the case of a *slowly varying envelope*, so that most of the **Fourier amplitude** $|A(k)|$ is peaked around some *characteristic value* of the wave number, say $k = k_0$. Then, for k *close* to k_0, we can use a **Taylor expansion** of the **dispersion relation** to first order in k, giving

$$\omega(k) \approx \omega(k_0) + (k - k_0)\, \left.\frac{\mathrm{d}\omega}{\mathrm{d}k}\right|_{k=k_0} + \dots \ , \tag{8.41}$$

so that

$$\omega(k) \approx \omega(k_0) + (k - k_0) c_g(k_0) + \dots \ . \tag{8.42}$$

Using this expression for $\omega = \omega(k)$ in the **Fourier integral** for the disturbance $\eta(t,x)$ gives

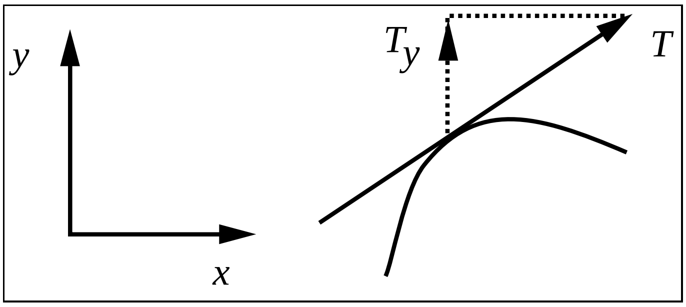
$$\eta(t,x) \approx \frac{1}{\sqrt{2\pi}}\, \mathrm{e}^{ik_0[x - \frac{\omega(k_0)}{k_0}t]} \int_{-\infty}^{\infty} A(k)\, \mathrm{e}^{i(k-k_0)(x-c_g t)}\, \mathrm{d}k \ . \tag{8.43}$$

The term in front of the integral represents a **carrier wave** that is purely **sinusoidal**, of wavenumber k_0 and with **phase speed** $c = \omega(k_0)/k_0$. The integral itself has a time dependence *only* through the term $(x - c_g t)$. This represents propagation of the **envelope** of the disturbance at a speed c_g, in other words, at the **group speed**.

8.4 Capillary waves

In considering **surface waves** on water we *assumed* that there was a **pressure balance** at the water–air interface. In other words, we assumed that there were no additional forces acting. However, we know that there is another force which acts at the surface of a fluid, namely **surface tension**. We are familiar with **surface tension** as the force that tends to make soap bubbles and water drops *spherical*. We shall see that *surface tension forces are proportional to the curvature of the fluid surface*. This means that **surface tension** *only* plays a role for **surface waves** when the **surface curvature** becomes dynamically important, which, for **small amplitude waves**, is in the limit of *small* **wavelength**. Thus **short wavelength waves**, or **ripples**, have a *different* **dispersion relation**, one controlled by **surface tension** effects. Such waves are known as **capillary waves**. For the case of water these waves have wavelengths less than a few centimetres.

At the **surface** of a fluid there is a **surface tension force** per *unit length*, $\boldsymbol{T}$, which is directed *tangential* to the surface, and which acts to flatten any "bumps" in the surface (just as the tension in a stretched rubber band). A precise definition is that at the **boundary** between two pieces of water surface, the two pieces of water surface pull on each other with an equal and opposite force in the direction *tangential* to the surface.



Since in a reference frame with **Cartesian coordinates** the **slope** of the surface is $\partial\eta/\partial x$, the force $\boldsymbol{T}$ has a *vertical component* which can be written, in the **small amplitude approximation**, as

$$T\,\frac{\partial\eta}{\partial x}\,. \tag{8.44}$$

This is true provided $T := |\boldsymbol{T}| \gg T_y$ where T_y is the vertical (upward) component. Now considering a small strip of surface (with x–size δx, and of unit length in z), which has a **surface tension force** at both ends, one can calculate the resultant **vertical force** due to **surface tension** as

$$T\,\frac{\partial\eta}{\partial x}\bigg|_{x+\delta x} - T\,\frac{\partial\eta}{\partial x}\bigg|_{x} \approx T\,\frac{\partial^2\eta}{\partial x^2}\,\delta x \tag{8.45}$$

Thus the **vertical force** per *unit area* is

$$T\,\frac{\partial^2\eta}{\partial x^2}\,, \tag{8.46}$$

and this must be balanced by a **pressure difference** between the **fluid pressure** p and the (constant) **atmospheric pressure** p_0. So we get

$$p - p_0 = T\,\frac{\partial^2\eta}{\partial x^2} \quad \text{at} \quad y = \eta(t,x)\,. \tag{8.47}$$

We can now use the **pressure condition** at the **free surface** as before, after **linearising**, to find

$$\frac{\partial \phi}{\partial t} + g\eta - \frac{t}{\rho}\frac{\partial^2 \eta}{\partial x^2} = 0 \quad \text{at} \quad y = \eta(t,x)\ . \tag{8.48}$$

Then we repeat the analysis for **surface gravity waves** as before. However, since we are seeking a solution of the form

$$\eta = A\cos(kx - \omega t)\ ,$$

we can simply replace g by $g + (Tk^2/\rho)$ in all relevant relations. This leads us directly to the **dispersion relation**

$$\omega^2 = gk + \frac{Tk^3}{\rho}\ , \tag{8.49}$$

so that

$$c = \left(\frac{g}{k} + \frac{Tk}{\rho}\right)^{1/2}\ , \quad c_g = \frac{g + 3Tk^2/\rho}{2\,(gk + Tk^3/\rho)^{1/2}}\ . \tag{8.50}$$

The above dispersion relation contains *both* the effects of **gravity** (operating at long wavelengths) and **surface tension** (operating at short wavelengths). For the case of **water**, **surface tension** effects actually dominate at wavelengths *below* about 1.7 cm. In this case the waves are known as **capillary waves** (capillary: hairlike). The dominance of surface tension effects can be measured through the *dimensionless* **parameter**

$$\frac{Tk^2}{\rho g}\ .$$

When this parameter is *large*, the **dispersion relation** can be approximated by

$$\omega^2 = \frac{Tk^3}{\rho}\ , \tag{8.51}$$

so that

$$c = \left(\frac{Tk}{\rho}\right)^{1/2}\ , \quad c_g = \frac{3}{2}\,c\ . \tag{8.52}$$

Thus, *short* wavelengths propagate *faster* than *long* wavelengths, which is *opposite* to the case for **surface gravity waves**. Also the **group speed** is *greater* than the **phase speed**, so wave crests move *backwards* through a wave packet as it propagates as a whole. The **frequency** of capillary waves is relatively high (above 70 Hz for $\lambda < 4$ mm), so that they can be excited by **acoustic noise**. (Thus it is **capillary waves** that can be seen in the glass of liquid on the top of a juke box!)

The behaviour of **capillary waves** can be seen when raindrops fall in water, exciting short wavelength waves. The pattern of wave crests from a raindrop falling in water and from a pebble are very different, reflecting the *different* **dispersion properties** caused by **surface tension** and **gravity**.

Wave crest pattern for rain drops and pebbles.

For **capillary–gravity waves**, when *both* effects are important, the **phase speed** has a *minimum*. This helps to explain the disturbance caused by a fishing line in a running stream (or a finger brushed over the surface of a bath!). If the flow has a speed in the right range, short wavelength waves will have a high **group speed** to propagate upstream against the flow, and so short wavelength waves will be seen in front of the narrow obstacle. On the other hand waves with slightly longer wavelengths will be swept downstream, so they form a pattern behind the obstacle.

Phase speed for capillary–gravity waves.

8.5 Sound waves

So far in this course we have concentrated almost entirely on **incompressible fluids** for which $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$ (with the exception of the equation of conservation of mass). **Sound waves**, on the other hand, result from the **compressibility** of a fluid, and so the **mass density** ρ will become another of the dynamical fluid variables, rather than simply a constant parameter.

We now turn to discuss the **governing equations** we wish to use. **Euler's equations** for an **inviscid flow** are still valid, since they were derived from the rate of change of linear momentum of a fluid element, and this is true even if the fluid is compressible. Hence, when the dynamical effects of **gravity** can be *neglected* (as we presently want to assume), we have

$$\rho \, \frac{\partial \boldsymbol{u}}{\partial t} + \rho \, (\boldsymbol{u} \cdot \boldsymbol{\nabla}) \boldsymbol{u} + \boldsymbol{\nabla} p = \boldsymbol{0} \, , \tag{8.53}$$

remembering that now $\rho = \rho(t, \boldsymbol{r})$. Even so, individual fluid elements still conserve their mass, and so we also have the **continuity equation**, which is

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0 \, . \tag{8.54}$$

As mentioned in the introduction to the course, if we consider a **compressible fluid**, we must have an **equation of state**, because, from a mathematical point of view, (8.53) and (8.54) provide us only with four equations for evolving five fluid variables, i.e., ρ, $\boldsymbol{u}$ (3!), and p. *The system of equations (8.53) and (8.54) by itself does not close.*

An **equation of state** will provide us with a fifth equation. In general this gives a functional relationship between, say, the **thermodynamical pressure** p, the **mass density** ρ, and the **specific entropy** s. Here we verge on the border of **fluid dynamics** with **thermodynamics**, which, amongst other aspects, studies relationships in **macroscopic physical systems** between such things as "**heat**" and "**work**". We will simply take the result that, if *heat conduction in a fluid is negligible*, then the appropriate equation of state is the so-called **adiabatic equation of state** given by

$$p \, \rho^{-\gamma} = \text{constant} \, , \tag{8.55}$$

where γ is a *dimensionless* characteristic constant for any particular fluid or gas. γ represents the ratio between the "**specific heats**" at constant pressure and constant volume, respectively. For example, for **air** $\gamma \approx 1.4$. Note that (8.55) does *not* depend on s, which reflects the assumption that heat conduction has been neglected.

The equation of state just given implies

$$\frac{\mathrm{D}}{\mathrm{D}t}(p \, \rho^{-\gamma}) = 0 \, ; \tag{8.56}$$

this expresses the fact that in **adiabatic flows** the **entropy** of *individual fluid elements* is conserved. The reason for employing an **adiabatic law** is that the variations of the **thermodynamical pressure** and of the **specific entropy** in **sound waves** are *too rapid* for **heat conduction** to play an important dynamical role.

8.5.1 Euler's equations in symmetric hyperbolic form

Let us start from the **adiabatic equation of state** (8.55) and write it as $p(\rho) = k\rho^{\gamma}$, with $k =$ constant. By the chain rule of differentiation we then have

$$\boldsymbol{\nabla} p = \frac{\mathrm{d}p(\rho)}{\mathrm{d}\rho} \, \boldsymbol{\nabla} \rho \, ,$$

which motivates to define a physical quantity

$$c_s^2(\rho) := \frac{\mathrm{d}p(\rho)}{\mathrm{d}\rho} = \gamma k \rho^{\gamma-1} = \gamma \, \frac{p(\rho)}{\rho} \, . \tag{8.57}$$

We can easily check that its physical dimension is $[\,\text{velocity}\,]^2$, or, in the MKS-system, $[\,\text{length}\,]^2\,[\,\text{time}\,]^{-2}$. The quantity $c_s = c_s(\rho)$ constitutes what is referred to as the **adiapatic speed of sound**.

When we now re-arrange terms conveniently, and make use of the **adiabatic equation of state** and, thus, (8.57), we can re-write (8.53) and (8.54) in the alternative form

$$\frac{c_s^2(\rho)}{\rho}\left(\frac{\partial\rho}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\rho\right) + c_s^2(\rho)\,\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0 \qquad (8.58)$$

$$\rho\,\frac{\partial\boldsymbol{u}}{\partial t} + \rho\,(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} + c_s^2(\rho)\,\boldsymbol{\nabla}\rho = \mathbf{0}\,. \qquad (8.59)$$

In a reference frame with **Cartesian coordinates**, this takes the explicit form

$$\mathsf{A}(\boldsymbol{X})\,\frac{\partial\boldsymbol{X}}{\partial t} + \mathsf{B}_x(\boldsymbol{X})\,\frac{\partial\boldsymbol{X}}{\partial x} + \mathsf{B}_y(\boldsymbol{X})\,\frac{\partial\boldsymbol{X}}{\partial y} + \mathsf{B}_z(\boldsymbol{X})\,\frac{\partial\boldsymbol{X}}{\partial z} = \mathbf{0}\,, \qquad (8.60)$$

for a **state vector** $\boldsymbol{X}$ defined by

$$\boldsymbol{X} := (\rho, u_x, u_y, u_z)^T\,,$$

and **coefficient matrices**

$$\mathsf{A}(\boldsymbol{X}) := \begin{bmatrix} c_s^2(\rho)/\rho & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 \\ 0 & 0 & \rho & 0 \\ 0 & 0 & 0 & \rho \end{bmatrix}\,,$$

$$\mathsf{B}_x(\boldsymbol{X}) := \begin{bmatrix} c_s^2(\rho)u_x/\rho & c_s^2(\rho) & 0 & 0 \\ c_s^2(\rho) & \rho u_x & 0 & 0 \\ 0 & 0 & \rho u_x & 0 \\ 0 & 0 & 0 & \rho u_x \end{bmatrix}\,,$$

$$\mathsf{B}_y(\boldsymbol{X}) := \begin{bmatrix} c_s^2(\rho)u_y/\rho & 0 & c_s^2(\rho) & 0 \\ 0 & \rho u_y & 0 & 0 \\ c_s^2(\rho) & 0 & \rho u_y & 0 \\ 0 & 0 & 0 & \rho u_y \end{bmatrix}\,,$$

$$\mathsf{B}_z(\boldsymbol{X}) := \begin{bmatrix} c_s^2(\rho)u_z/\rho & 0 & 0 & c_s^2(\rho) \\ 0 & \rho u_z & 0 & 0 \\ 0 & 0 & \rho u_z & 0 \\ c_s^2(\rho) & 0 & 0 & \rho u_z \end{bmatrix}\,.$$

The system of equations (8.60), which constitutes a coupled system of non-linear partial differential equations of first order, is said to be of **symmetric hyperbolic form**: **symmetric** because each of the four coefficient matrices A, B_x, B_y and B_z is a *symmetric* matrix, **hyperbolic** because each of B_x, B_y and B_z has *generalised* eigenvalues with respect to A (instead of the unit matrix

1) which are *real-valued*. For a matrix $\mathsf{B}_{i=x,y,z}$, the corresponding so-called **characteristic polynomial** reads

$$0 = \det(\mathsf{B}_i - \lambda\,\mathsf{A}) = c_s^2\,\rho^2\,(\lambda - u_i)^2\,[\,\lambda - (u_i + c_s)\,]\,[\,\lambda - (u_i - c_s)\,]\ , \quad (8.61)$$

so that the so-called **characteristic speeds** (generalised eigenvalues) of the system of equations (8.60) are given by

$$|\,\lambda\,| \in \{\,|\,u_i\,|,\ |\,u_i + c_s(\rho)\,|,\ |\,u_i - c_s(\rho)\,|\,\}\ .$$

This result implies that relative to an inviscid flow (with flow velocity $\boldsymbol{u}$) disturbances in the values of the fluid variables are either (i) dragged along with the flow, $|\,\lambda - u_i\,| = 0$, or (ii) propagate at the **adiabatic speed of sound**, $|\,\lambda - u_i\,| = c_s(\rho)$.

Symmetric hyperbolic systems have been introduced into **mathematical physics** by the German mathematical physicist Kurt Otto Friedrichs (1901–1982).[2] The really interesting aspect of such systems, which is of immense practical value, is the fact that for this class of partial differential equations **theorems** on the **existence**, **uniqueness** and **stability** of **solutions** have been successfully established. In particular, **Cauchy's initial value problem** for a specific physical problem is **well-posed** when its governing equations can be cast into **symmetric hyperbolic form**.

8.5.2 Small-amplitude sound waves: linearisation

We now investigate **small amplitude sound waves** in an **inviscid fluid**, *assuming* fluctuations around an **equilibrium configuration** characterised by vanishing **flow velocity**,

$$\boldsymbol{u}_0 = \boldsymbol{0}\ ,$$

a constant **thermodynamical pressure**, p_0, and a constant **mass density**, ρ_0. Then the **flow velocity**, **thermodynamical pressure** and **mass density** of the **perturbed fluid state** will be given by (including fluctuation terms labelled by a subscript 1)

$$\boldsymbol{u} = \boldsymbol{u}_1\ , \qquad \rho = \rho_0 + \rho_1\ , \qquad p = p_0 + p_1\ , \qquad (8.62)$$

with

$$\frac{\rho_1}{\rho_0} \ll 1\ , \qquad \frac{p_1}{p_0} \ll 1\ .$$

We will *linearise* the **governing equations** (8.53), (8.54) and (8.55) by *neglecting* any terms that are of *quadratic order* (or higher) in the fluctuation quantities $\boldsymbol{u}_1$, p_1 and ρ_1 (so, e.g., we will neglect a term $\rho_1\boldsymbol{u}_1$).

From the **adiabatic equation of state** and (8.56) we have that the product $p\,\rho^{-\gamma}$ is a conserved quantity for each fluid element. As initially (in the equilibrium state) the value of this product was $p_0\,\rho_0^{-\gamma}$ for *all* fluid elements, $p\,\rho^{-\gamma}$ must be equal to $p_0\,\rho_0^{-\gamma}$ *everywhere*. That is,

$$(p_0 + p_1)\,(\rho_0 + \rho_1)^{-\gamma} = p_0\,\rho_0^{-\gamma}\ , \qquad (8.63)$$

[2]Of course, there are symmetric hyperbolic systems that are *linear*, too.

leading to

$$\left(1+\frac{p_1}{p_0}\right)\left(1+\frac{\rho_1}{\rho_0}\right)^{-\gamma} = 1\ . \tag{8.64}$$

Therefore, expanding in powers of the *small* dimensionless quantity ρ_1/ρ_0, we have

$$\left(1+\frac{p_1}{p_0}\right)\left(1-\gamma\,\frac{\rho_1}{\rho_0}+\dots\right) = 1\ , \tag{8.65}$$

so that *linearising* this relation leads to

$$\frac{p_1}{p_0} = \gamma\,\frac{\rho_1}{\rho_0}\ . \tag{8.66}$$

Alternatively, this can be written as

$$p_1 = c_s^2\,\rho_1 \qquad \text{where} \qquad c_s := (\gamma p_0/\rho_0)^{1/2}\ . \tag{8.67}$$

Different to (8.57), we can presently refer to the quantity c_s as the **isentropic speed of sound** of the particular fluid medium considered because it is a *constant*. For still **air** at sea level c_s is about 330 m s^{-1} to 340 m s^{-1}, while for **water** we have c_s between 1400 m s^{-1} and 1450 m s^{-1}.

Linearising **Euler's equations** (8.53) according to the scheme outlined, we get

$$\rho_0\,\frac{\partial \boldsymbol{u}_1}{\partial t} = -\,\boldsymbol{\nabla} p_1\ , \tag{8.68}$$

while *linearising* the **continuity equation** (8.54) yields

$$\frac{\partial \rho_1}{\partial t} + \rho_0\,\boldsymbol{\nabla}\cdot\boldsymbol{u}_1 = 0\ . \tag{8.69}$$

Now taking the divergence of (8.68) results in

$$\rho_0\,\frac{\partial \boldsymbol{\nabla}\cdot\boldsymbol{u}_1}{\partial t} = -\,(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})p_1\ , \tag{8.70}$$

so that substitution of $\boldsymbol{\nabla}\cdot\boldsymbol{u}_1$ from (8.69), using $p_1 = c_s^2\,\rho_1$, finally gives a linear partial differential equation that describes variations of p_1 both in time and space. This equation is given by

$$-\,\frac{1}{c_s^2}\,\frac{\partial^2 p_1}{\partial t^2} + (\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})p_1 = 0\ . \tag{8.71}$$

Thus, the **pressure fluctuations** p_1, and also the other fluctuation quantities (ρ_1 and $\boldsymbol{u}_1$), all satisfy the classical **wave equation**, here in its three-dimensional realisation.

In the case of **plane symmetric pressure waves**, i.e., when the physical properties of a wave are constant along the directions tangent to a family of plane surfaces (so that the waves are effectively one-dimensional), we have to solve the **wave equation**

$$-\,\frac{1}{c_s^2}\,\frac{\partial^2 p_1}{\partial t^2} + \frac{\partial^2 p_1}{\partial z^2} = 0\ . \tag{8.72}$$

In this case, the **general solution** is given by

$$p_1(t,z) = f(z - c_s t) + g(z + c_s t) \ , \tag{8.73}$$

with f and g *arbitrary* real-valued functions of t and z that are twice continuously differentiable. In a specific application we will have to adapt f and g to *given* **initial** and **boundary conditions**.

Let us verify the statement on the general solution to the **wave equation** in the **plane symmetric case**. Introducing the **auxiliary independent variables**

$$u := z - c_s t \ , \qquad v := z + c_s t \ ,$$

we have by the chain rule of differentiation

$$\frac{\partial p_1}{\partial t} = \frac{\partial f}{\partial u} \times (-c_s) + \frac{\partial g}{\partial v} \times c_s \ , \qquad \frac{\partial^2 p_1}{\partial t^2} = \frac{\partial^2 f}{\partial u^2} \times (-c_s)^2 + \frac{\partial^2 g}{\partial v^2} \times c_s^2 \ ,$$

and

$$\frac{\partial p_1}{\partial z} = \frac{\partial f}{\partial u} \times 1 + \frac{\partial g}{\partial v} \times 1 \ , \qquad \frac{\partial^2 p_1}{\partial z^2} = \frac{\partial^2 f}{\partial u^2} \times 1^2 + \frac{\partial^2 g}{\partial v^2} \times 1^2 \ ,$$

so that substitution into (8.72) leads to identical cancellations.

The two terms in the general solution of the **wave equation** in 1-D correspond to wave disturbances propagating at speed c_s to the **right** [represented by $f(z - c_s t)$] and to the **left** [represented by $g(z + c_s t)$], *without change of shape*. Thus, we see that within our approximation scheme **small amplitude sound waves** are **non-dispersive**!

For a **small amplitude wave** that is **spherically symmetric**, so that $p_1 = p_1(t,r)$ only, the **wave equation** becomes

$$-\frac{1}{c_s^2}\frac{\partial^2 p_1}{\partial t^2} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial p_1}{\partial r}\right) = 0 \ . \tag{8.74}$$

This can be simplified by making the substitution

$$p_1 = \frac{h(t,r)}{r} \ .$$

Then it follows that

$$-\frac{1}{c_s^2}\frac{\partial^2 h}{\partial t^2} + \frac{\partial^2 h}{\partial r^2} = 0 \ . \tag{8.75}$$

This equation is of the *same* form as the **wave equation** in the **plane symmetric case** discussed before, so that the **general solution** is simply given by

$$p_1(t,r) = \frac{1}{r}\,[\,F(r - c_s t) + G(r + c_s t)\,] \ . \tag{8.76}$$

The arbitrary functions in the general solution will have to be chosen so as to suit the **initial** and **boundary conditions** of a *given problem*, e.g., sinusoidal functions, or linear combinations of them. If it is known that a **source** of **sound radiation** is at a specific **location**, then one might impose a so-called **radiation condition**. In such a case one *only* selects solutions that describe **outward propagating waves**, i.e., solutions that depend on $(r - c_s t)$ only.